The Handbook of MATH221

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A disclaimer to the user. This handbook is meant to act as a supplemental resource for MATH221 (Business Calculus) taught at the University of Delaware. In this booklet you will find the major equations and topics of the course, each paired with an example or two. This is neither meant to replace the designated course textbook nor lecture notes. However, it *will* act as a quick reference when studying and completing homework assignments. For those wishing to use it for other similar business calculus courses, please note that the notation might differ, trigonometric functions are not covered, and integration by parts is omitted. The author wishes to thank his fellow graduate students for their invaluable input and critiques. If the reader finds any typos or mistakes in the text, please email the author at nakaoj@udel.edu.

Contents

The (First) Derivative

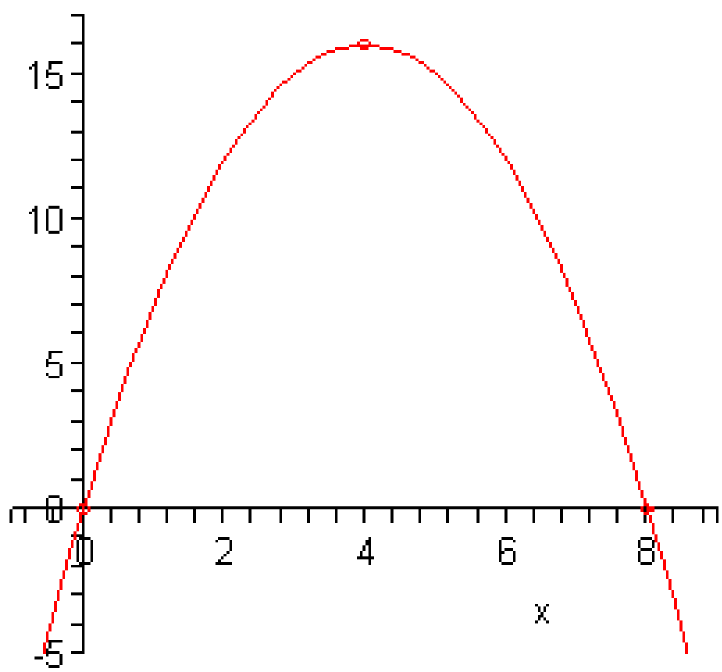
What is the derivative of a function? It is the rate of change of that function at any value in the function's domain. In other words, it tells us if the function is *increasing* or *decreasing*.
Notation: The derivative of a function $f(x)$ is typically denoted in two ways:

$$f'(x) \qquad \text{and} \qquad \frac{df}{dx} \text{ (Leibniz notation *see Appendix F)}$$

Example 1.1. Let $f(x)$ be the amount of money (in dollars) made from baking x cakes. Then, $f'(x)$ is tells us the rate at which we are either making or losing money. For instance, if $f'(40) = -10$, then this means that when we bake 40 cakes, we are *losing* money at a rate of 10 dollars/cake. Whereas, if $f'(20) = 24$, then this means that when we bake 20 cakes, we are *making* money at a rate of 24 dollars/cake.

Notice that when $f'(x) < 0$ (i.e., negative), it means that we are losing money; when $f'(x) > 0$ (i.e., positive), it means that we are making money. **please refer to Appendix G for how to find the correct units.*

Example 1.2.



Let $f(x)$ be the parabola plotted above. Observe that $f(x)$ is *increasing* for $x < 4$, *decreasing* for $x > 4$, and *neither increasing nor decreasing* at $x = 4$. Thus, we write

$$f'(x) > 0 \text{ for } x < 4$$
$$f'(x) < 0 \text{ for } x > 4$$
$$f'(4) = 0$$

The 3 “Big Ones” of Differentiation

For the sake of this course, we only need to know the derivatives of three basic functions.

Polynomials, $f(x) = x^n$

$$\frac{df}{dx} = nx^{n-1}$$

Exponentials, $f(x) = e^x$

$$\frac{df}{dx} = e^x$$

Natural Logarithm, $f(x) = \ln(x)$

$$\frac{df}{dx} = \frac{1}{x}$$

**note: the derivative of any constant k is zero.*
Moreover, the following two rules apply:

1. You can “add” derivatives: $\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{df}{dx} + \frac{dg}{dx}$

2. You can “scale” derivatives: $\frac{d}{dx}\big(k \cdot f(x)\big) = k\frac{df}{dx}$

Example 2.1. Find the derivatives of the following functions:

(a) $f(x) = x^{8.1} + 4x^3$, (b) $f(x) = 5e^x$, (c) $f(x) = \ln x^4$, (d) $f(x) = (x-1)^2$

(a) The two laws tell us that we can simply add the derivatives of $x^{8.1}$ and $4x^3$.

$$\frac{d}{dx}\big(x^{8.1} + 4x^3\big) = \frac{d}{dx}(x^{8.1}) + \frac{d}{dx}(4x^3) = \frac{d}{dx}x^{8.1} + 4\frac{d}{dx}x^3 = 8.1x^{7.1} + 4 \cdot 3x^2 = 8.1x^{7.1} + 12x^2$$

(b) The second law tells us we can simply “move” the 5 to the front.

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x = 5e^x$$

(c) Notice that $f(x) = \ln x^4$ is not in a form of one of the “big ones.” Thus, we must rewrite $f(x)$ so that we have one of the three forms! Simply use laws of logarithms (**see Appendix D*). Rewriting, $f(x) = \ln x^4 = 4\ln x$. Now, we can take a derivative.

$$\frac{d}{dx}\ln x^4 = \frac{d}{dx}(4\ln x) = 4\frac{d}{dx}\ln x = \frac{4}{x}$$

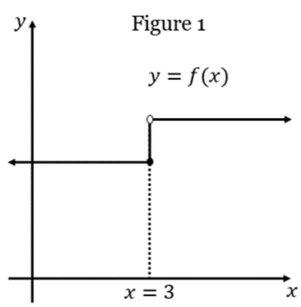
(d) Again, $f(x) = (x-1)^2$ is not in one of the three basic forms. So, we must expand/FOIL as $f(x) = (x-1)^2 = x^2 - 2x + 1$. Now it is the sum of three easy functions! **note: problems will not always be this easy; see product/quotient/chain rules and logarithmic differentiation.*

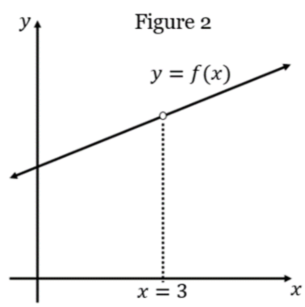
$$\frac{d}{dx}(x-1)^2 = \frac{d}{dx}(x^2 - 2x + 1) = 2x - 2 + 0 = 2x - 2$$

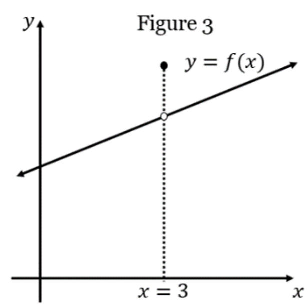
Limits

What is the limit of a function at a value $x = a$? It is the $y-$value the function approaches as x gets close to a. As seen in figures 2 and 3 below, notice that even if there is a hole at $x = 3$, the function is still approaching a $y-$value.
note: the limit at $x = 3$ does not exist (DNE) in figure 1 since the limit from the **left does not match the limit from the **right**; the left and right limits **must be equal**!*
When does the limit not exist? For the sake of this course, there are only two situations where the limit does not exist. As already mentioned, the first situation is if the left and right limits are not equal, as in figure 1. The second situation is if the limit is ∞, since ∞ is not technically a number. Pretty much, if you ever get ∞ as an answer in this course, you should be very concerned...
Notation: If a function $f(x)$ approaches the $y-$value of L as x approaches a, we write

$$\lim_{x \to a} f(x) = L.$$







Example 3.1. Find the limits of the following functions at $x = 2$ if they exist. Otherwise, show if the limit DNE.

$$\text{(a) } f(x) = x^3 - 4x + 1, \quad \text{(b) } f(x) = \frac{x^2 - 4}{x - 2}, \quad \text{(c) } f(x) = \frac{x^2 - 1}{x - 2}, \quad \text{(d) } f(x) = \ln(x - 2)$$

(a)

$$\lim_{x \to 2} x^3 - 4x + 1 = 2^3 - 4(2) + 1 = 1.$$

(b)

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}?$$

OKAY! So we got $\frac{0}{0}$, but this might still be fine. Whenever we get $\frac{0}{0}$, it is possible there could be something we can factor and cancel out. In fact, we can since

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

(c)

$$\lim_{x \to 2} \frac{x^2 - 1}{x - 2} = \frac{3}{0} = \infty.$$

Unlike in part (b), we did not get $\frac{0}{0}$, so there was no hope of finding something to factor. For the sake of this course, if you get a nonzero number divided by zero, then the limit approaches infinity and DNE.

(d)

$$\lim_{x\to 2} \ln(x-2) = -\infty.$$

Clearly, the limit DNE since we get $-\infty$.

Asymptotes and Holes

Vertical Asymptotes and Holes: There are two simple steps!

Step 1: Find all the $x-$values where the function is undefined. Typically, this means when does is the denominator zero. However, this is not always true (e.g., $\ln x$ is undefined at $x = 0$, yet this is not a fraction).
Step 2: The take limit at each of the values found in step 1. If the limit exists, then it is a hole. If the limit approaches ∞, then it is a vertical asymptote.

Example 4.1. Find the vertical asymptote(s) and/or hole(s) of the following functions.

$$\text{(a) } f(x) = \frac{x-1}{(x-1)(x+2)}, \quad \text{(b) } f(x) = \begin{cases} x-2, & x < 3 \\ 1, & 3 < x \leq 5 \\ \ln(x-5), & 5 < x \end{cases}$$

(a) The function $f(x)$ is clearly undefined at $x = 1$ and $x = -2$. One can easily see that the limit exists at $x = 1$ (the limit is $\frac{1}{3}$). However, the limit is $-\infty$ at $x = -2$. Thus, there is a **hole** at $\left(1, \frac{1}{3}\right)$ and a vertical asymptote $x = -2$.
**note: see the previous section on limits if the limits were not clear.*

(b) The function $f(x)$ is clearly undefined at $x = 3$ since the inequalities are $<$ and $>$, rather than $\leq$ and $\geq$; and the function $f(x)$ is undefined at $x = 5$ since $\ln 0 = -\infty$.
The left and right limits at $x = 3$ are the same, so there is a hole at $(3, 1)$.

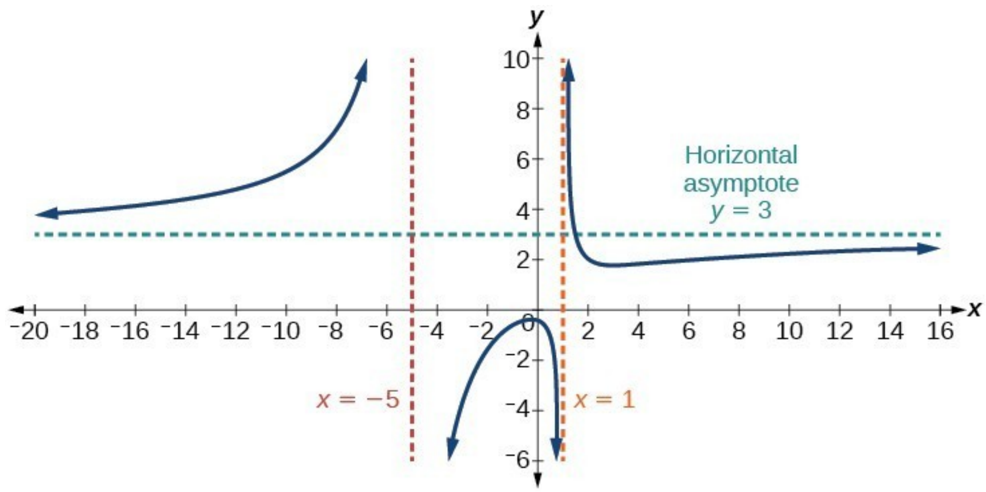
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 1$$

**note: if the left and right limits for a piecewise function are unclear, please see Appendix I.*

Since the right limit at $x = 5$ is $-\infty$, there is a vertical asymptote $x = 5$.

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \ln(x-5) = \ln 0 = -\infty.$$

Horizontal Asymptotes: We simply just want to take the limit as $x \to \infty$ and $x \to -\infty$. In other words, as x goes REALLY far out, will the function $f(x)$ eventually "level out?" (the figure below paints a better picture).



In the figure above, notice that as $x \to \infty$ and $x \to -\infty$, the function "levels out" at the $y-$value 3. As such, the only horizontal asymptote is the line $y = 3$.
**note: in some cases there are two horizontal asymptotes; just imagine the limit is different for $x \to \infty$ and $x \to -\infty$.*

Now, we go through two typical types of problems we encounter in this course. Find the horizontal asymptote(s) of the following functions, if possible.

Example 4.2. When you have a polynomial divided by another polynomial. The way to solve these types of problems is to divide the numerator and denominator by the *largest power*.

$$\text{(a) } f(x) = \frac{4x^4 - 3x^2 - 4}{x^4 + 3}, \quad \text{(b) } f(x) = \frac{x^6 - 3x^2 + 1}{x^3 - 2x + 1}, \quad \text{(c) } f(x) = \frac{x^3 - 2}{x^6 + 2x^5 - 2x + 7}$$

(a) The largest power is x^4. So we will divide the numerator and denominator by x^4.

$$\lim_{x\to\infty} \frac{4x^4 - 3x^2 - 4}{x^4 + 3} = \lim_{x\to\infty} \frac{4x^4 - 3x^2 - 4}{x^4 + 3} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to\infty} \frac{4 - \frac{3}{x^2} - \frac{4}{x^4}}{1 + \frac{3}{x^4}} = \frac{4}{1} = 4$$

Since the limit exists, the horizontal asymptote is $y = 4$. It turns out that if you take the limit as $x \to -\infty$, you get the same horizontal asymptote.

(b) The largest power is x^6. So we will divide the numerator and denominator by x^6.

$$\lim_{x\to\infty} \frac{x^6 - 3x^2 + 1}{x^3 - 2x + 1} = \lim_{x\to\infty} \frac{x^6 - 3x^2 + 1}{x^3 - 2x + 1} \cdot \frac{\frac{1}{x^6}}{\frac{1}{x^6}} = \lim_{x\to\infty} \frac{1 - \frac{3}{x^4} + \frac{1}{x^6}}{\frac{1}{x^3} - \frac{2}{x^5} + \frac{1}{x^6}} = \frac{1}{0} = \infty.$$

Since the limit DNE, there does not exist a horizontal asymptote. You get the same if you take the limit as $x \to -\infty$.

(c) The largest power is x^6. So we will divide the numerator and denominator by x^6.

$$\lim_{x\to\infty} \frac{x^3 - 2}{x^6 + 2x^5 - 2x + 7} = \lim_{x\to\infty} \frac{x^3 - 2}{x^6 + 2x^5 - 2x + 7} \cdot \frac{\frac{1}{x^6}}{\frac{1}{x^6}} = \lim_{x\to\infty} \frac{\frac{1}{x^6} - \frac{2}{x^6}}{1 + \frac{2}{x} - \frac{2}{x^5} + \frac{7}{x^6}} = \frac{0}{1} = 0$$

Since the limit exists, the horizontal asymptote is $y = 0$. It turns out that if you take the limit as $x \to -\infty$, you get the same horizontal asymptote.

Example 4.3. When you are dealing with exponentials

$$\text{(a) } f(x) = e^{2x}, \quad \text{(b) } f(x) = \frac{e^x}{e^x + e^{-x}}$$

(a)

$$\lim_{x\to\infty} e^{2x} = e^{\infty} = \infty$$
$$\lim_{x\to-\infty} e^{2x} = e^{-\infty} = 0$$

**note: recall the graph of* e^x. There only exists a horizontal asymptote when $x \to -\infty$. So, the only horizontal asymptote is $y = 0$.

(b) Similar to the previous example with polynomials, we will divide the numerator and denominator by the exponential causing us trouble.
For the limit $x \to \infty$, observe that e^x causes us trouble since $e^{+\infty} = \infty$. So we will divide the numerator and denominator by e^x.

$$\lim_{x\to\infty} \frac{e^x}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x + e^{-x}} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + e^{-\infty}} = \frac{1}{1+0} = 1$$

For the limit $x \to -\infty$, observe that e^{-x} causes us trouble since $e^{-(-\infty)} = e^{\infty} = \infty$. So we will divide the numerator and denominator by e^{-x}.

$$\lim_{x\to-\infty} \frac{e^x}{e^x + e^{-x}} = \lim_{x\to-\infty} \frac{e^x}{e^x + e^{-x}} \cdot \frac{\frac{1}{e^{-x}}}{\frac{1}{e^{-x}}} = \lim_{x\to-\infty} \frac{e^{2x}}{e^{2x} + 1} = \frac{e^{-\infty}}{e^{-\infty} + 1} = \frac{0}{0+1} = 0$$

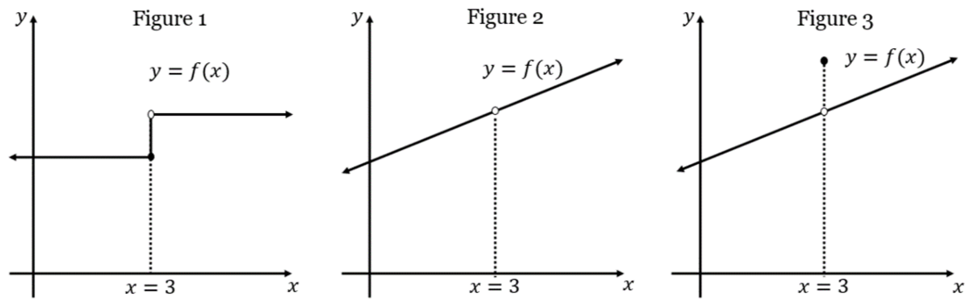
So, the two horizontal asymptotes are $y = 1$ and $y = 0$.

Continuity and Differentiability

Continuity: A function $f(x)$ is continuous at $x = a$ if three conditions hold:

1. $f(a)$ **is defined, i.e.,** $f(x)$ **has a point at** $x = a$.
2. $\lim_{x\to a} f(x)$ **exists, i.e., the limit exists.**
3. $f(a) = \lim_{x\to a} f(x)$**, i.e., if (1) and (2) are true, then their values are the same**.

The figure below shows examples of when each of the three conditions breaks.
- Figure 1: (1) is true since there is an actual point $(3, f(3))$. (2) is not true since the limit DNE at $x = 3$.
**note: we don't even talk about condition (3) unless (1) and (2) are both true.*
- Figure 2: (2) is true since the left and right limits both exist and approach the same $y-$value. (1) is clearly not true since there is not an actual point at $x = 3$.
- Figure 3: (1) is true since there is an actual point $(3, f(3))$. (2) is clearly true since the left and right limits both exist and approach the same $y-$value. However, (3) fails since these are not the same value.



Example 5.1. Are the following functions continuous at the point when $x = 1$?

$$\text{(a) } f(x) = x^2 - 1,\ x \neq 1, \quad \text{(b) } f(x) = \begin{cases} x^2 - 1, & x \leq 1, \\ x - 1, & 1 \leq x. \end{cases}, \quad \text{(c) } f(x) = \begin{cases} x^2 - 1, & x \leq 1, \\ x, & 1 < x. \end{cases}$$

(a)
Condition (1): False. The function only holds when $x \neq 1$.
Not continuous.

(b)
Condition (1): True. $f(1) = 0$.
Condition (2): True. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 0$.
Condition (3): True. (1)=(2).
Is Continuous.

(c)
Condition (1): True. $f(1) = 0$.
Condition (2): False. $\lim_{x \to 1^-} f(x) = 0 \neq 1 = \lim_{x \to 1^+} f(x)$.
Not continuous.

Differentiability: For the sake of this course, a function $f(x)$ is differentiable at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists. Or rather, $f(x)$ is *not* differentiable at $x = a$ if the *slopes* from the left and right are different, $f(a)$ DNE, or a jump discontinuity occurs.
**note: recall that the slope at $x = a$ is just the derivative at that value.*

Now that we know how to determine continuity and differentiability formally, we introduce two very powerful theorems/laws.

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

In other words, if you already know that $f(x)$ is differentiable, then there is no need to check for continuity. Similarly, if you already know that $f(x)$ is not continuous, then there is no need to check for differentiability.

Example 5.2. Are the same functions as before differentiable at the point when $x = 1$?

$$\text{(a) } f(x) = x^2 - 1,\ x \neq 1, \quad \text{(b) } f(x) = \begin{cases} x^2 - 1, & x \leq 1, \\ x - 1, & 1 \leq x. \end{cases}, \quad \text{(c) } f(x) = \begin{cases} x^2 - 1, & x \leq 1, \\ x, & 1 < x. \end{cases}$$

(a) No. By one of the two theorems, $f(x)$ is not continuous at $x = 1$, thus $f(x)$ is also not differentiable at $x = 1$.

(b) No. Although continuous at $x = 1$, the slope from the *left* of $x = 1$ is $2x\big|_{x=1} = 2$. Whereas, the slope from the *right* of $x = 1$ is 1.

(c) No. By one of the two theorems, $f(x)$ is not continuous at $x = 1$, thus $f(x)$ is also not differentiable at $x = 1$.

The Limit Definition of the Derivative

We learned the derivatives of the three "big ones" in a previous section. However, those were just formulas/rules we simply followed. Now, we give the *formal* way to find the derivative of a function $f(x)$. The *limit definition of the derivative* is defined as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

This is nothing more than a "plug and chug" computation. The only goal is to **get rid of the h in the denominator!** Obviously, if we plug in $h = 0$, we get zero in the denominator. Thus, we should want to get rid of the h since that is what is causing so much trouble. Here are examples for the three types of problems we will encounter in this course.

Example 6.1. Polynomials. This is a simple "plug and chug," nothing more. Stuff should cancel.

$$f(x) = x^2 - 4x + 1$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\left[(x+h)^2 - 4(x+h) + 1\right] - \left[x^2 - 4x + 1\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[x^2 + 2xh + h^2 - 4x - 4h + 1\right] - \left[x^2 - 4x + 1\right]}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h - 4)}{h} \\ &= \lim_{h \to 0} 2x + h - 4 \\ &= 2x - 4. \text{ *once you take the limit, you NO LONGER write } \lim_{h \to 0} \end{aligned}$$

Example 6.2. Fractions. You just need to find a common denominator.

$$f(x) = \frac{1}{x}$$

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h}\cdot\frac{x}{x} - \frac{1}{x}\cdot\frac{x+h}{x+h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2}
\end{aligned}$$

Example 6.3. Radicals/square roots. You need to multiply by the conjugate.

$$f(x) = \sqrt{9-x}$$

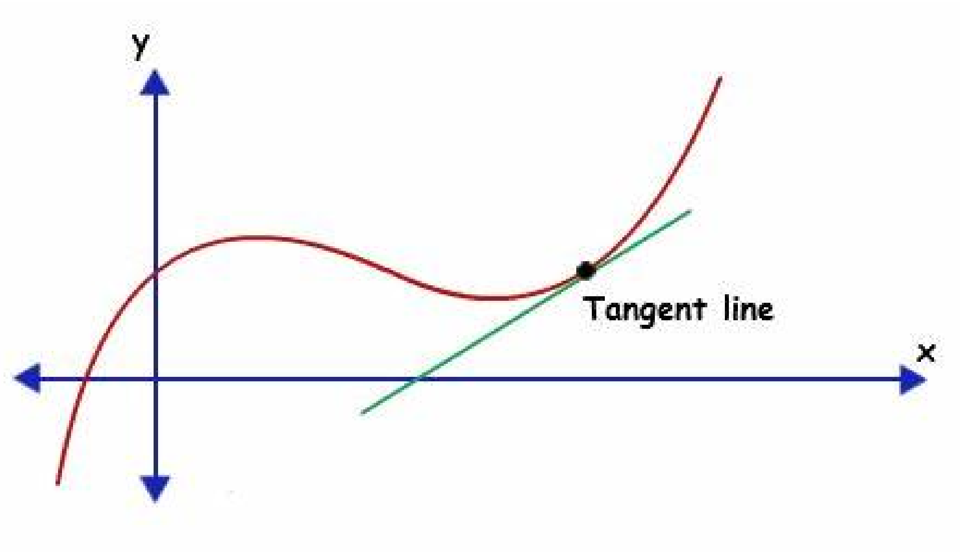
$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0} \frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h} \cdot \frac{\sqrt{9-(x+h)} + \sqrt{9-x}}{\sqrt{9-(x+h)} + \sqrt{9-x}} \\
&= \lim_{h\to 0} \frac{(9-(x+h)) - (9-x)}{h(\sqrt{9-(x+h)} + \sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-h}{h(\sqrt{9-(x+h)} + \sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)} + \sqrt{9-x}} \\
&= \frac{-1}{2\sqrt{9-x}} \\
&= \frac{-1}{2}(9-x)^{-1/2}.
\end{aligned}$$

Tangent Lines

As seen in the figure below, the tangent line is the line *tangent* to the curve. If we want to know the tangent line equation of a function $f(x)$ (centered) at the point $x = a$, we simply use the point-slope form. Recall that the slope of a graph is just the derivative at that point. Thus, the tangent line equation is

$$\boxed{y - f(a) = f'(a)(x - a)}$$

Note that any point (x, y) we plug into the tangent line equation lies on the line, as shown in the figure below.



Example 7.1. Find the tangent line for $f(x) = x^4 - 3x^2 + 1$ at the point where $x = 1$.

We need three pieces of information: a, $f(a)$, and $f'(a)$. We know that $a = 1$. We can easily find $f(a) = f(1) = 1^4 - 3 \cdot 1^2 + 1 = -1$. And, lastly,

$$f'(1) = \frac{d}{dx}(x^4 - 3x^2 + 1)\big|_{x=1} = (4x^3 - 6x)\big|_{x=1} = -2.$$

Thus, the tangent line equation desired is

$$y + 1 = -2(x - 1)$$

Example 7.2. If we know that the tangent line of a function $f(x)$ goes through the points $(0, 2)$ and $(2, 6)$. What is the tangent line at the point where $x = 1$?

We need three pieces of information: a, $f(a)$, and $f'(a)$. We know that $a = 1$. We also know that $f'(a) = f'(1)$ is the slope, which we can compute from the two points, i.e., "rise over run."

$$f'(1) = \frac{6 - 2}{2 - 0} = 2$$

We just need $f(a) = f(1)$. Well, we know the tangent line equation so far is

$$y - f(1) = 2(x - 1)$$

Thus, since we know that $(0, 2)$ and $(2, 6)$ lie *on* the tangent line, I can simply plug either of those points into the equation and get $f(1)$. Plugging in the point $(0, 2)$,

$$2 - f(1) = 2(0 - 1) \quad \Rightarrow \quad f(1) = 4.$$

So, the tangent line equation desired is

$$y - 4 = 2(x - 1)$$

Linear Approximation

Idea: Use the tangent line to approximate $f(x)$. Recall the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

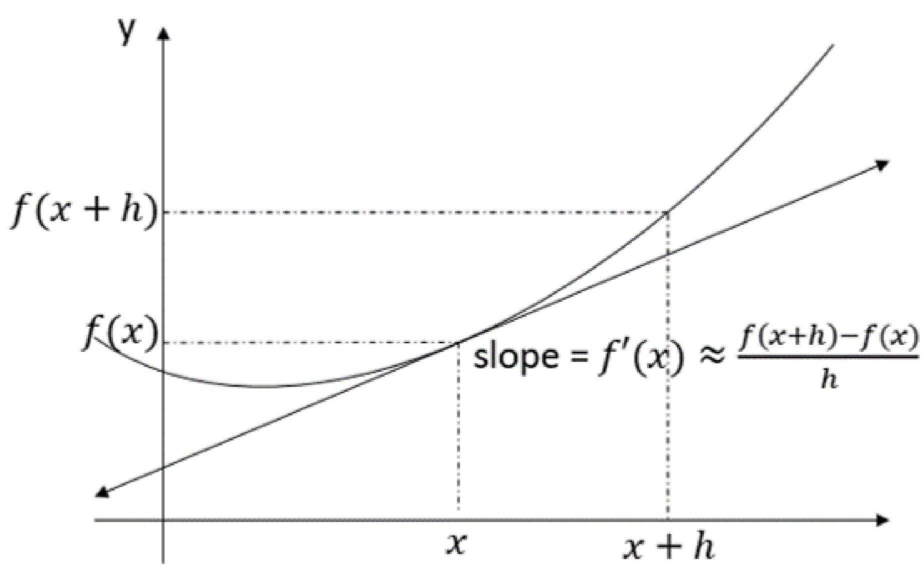
where $x + h$ is a (small) step away from x, with step-size h. Well, if we don't let $h \to 0$, then the difference quotient in the limit definition of the derivative says

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

which can be rewritten as

$$\boxed{f(x+h) \approx f(x) + hf'(x)}.$$

In other words, if I know $f(x)$ and $f'(x)$, then I can go out a small distance h to the right to approximate/predict $f(x + h)$, as seen in the figure below.
**note: the linear approximation equation just comes from the limit definition of the derivative, so there's no need to "memorize" another formula.*



Example 8.1. Let $s(t) = -0.1t^2 - 10t + 50$ be the height (in meters) of a baseball thrown in the air at time t (in seconds). Approximate the height of the ball at $t = 2.1$ seconds.

With a moment's thought, we can see that the known time is $t = 2$ and the step-size is $h = 0.1$ seconds, i.e., $t + h = 2.1$. All we need is $s(2)$ and $s'(2)$.

$$s(2) = -0.1(2)^2 - 10(2) + 50 = -0.4 - 20 + 50 = 29.6 \text{ meters}.$$

$$s'(2) = [-0.2t - 10]_{t=2} = -0.4 - 10 = -10.4 \text{ meters per second.}$$

So, using the linear approximation equation

$$s(2.1) \approx s(2) + 0.1s'(2) = 29.6 + (0.1)(-10.4) = 29.6 - 1.04 = 28.56 \text{ meters.}$$

*note: if you have a hard time figuring out the units, please see Appendix G.

Example 8.2. Approximate the value of $\sqrt{4.01}$.

Recall the linear approximation equation,

$$f(x+h) \approx f(x) + h \cdot f'(x)$$

Letting $f(x) = \sqrt{x}$, $x = 4$ and $h = 0.01$,

$$\sqrt{4.01} = f(4.01) \approx f(4) + 0.01f'(4) = \sqrt{4} + 0.01 \cdot \frac{1}{2\sqrt{4}} = 2 + \frac{0.01}{4} = 2 + \frac{1}{400} = 2.0025$$

Chain Rule

We have learned how to take a derivative of functions that are of the form x^n, e^x, and $\ln x$. However, now we learn how to take derivatives of more complicated functions, particularly those that have the general form $f\big(g(x)\big)$ (i.e., a function of a function). The **general chain rule** states

$$\boxed{\frac{d}{dx}\big[f\big(g(x)\big)\big] = \frac{df}{dg} \cdot \frac{dg}{dx}}$$

...or, in a simpler albeit deceiving form, $\big(f(g(x))\big)' = f'\big(g(x)\big) \cdot g'(x)$.
The chain rules for the three "big ones" are

Polynomials, $f(x) = x^n$

$$\frac{d}{dx}\big[(g(x))^n\big] = (g(x))^{n-1} \cdot g'(x)$$

Exponentials, $f(x) = e^x$

$$\frac{d}{dx}\big[e^{g(x)}\big] = e^{g(x)} \cdot g'(x)$$

Natural Logarithm, $f(x) = \ln(x)$

$$\frac{d}{dx}\big[\ln g(x)\big] = \frac{1}{g(x)} \cdot g'(x)$$

Example 9.1. Compute the derivatives of the following functions.

$$\text{(a) } f(x) = (2x^2 - 4x + 5)^{4.1}, \quad \text{(b) } f(x) = e^{x^4 - 3x + 1}, \quad \text{(c) } f(x) = \ln(4x^3 - 4x + 7)$$

(a)
$$f'(x) = 4.1(2x^2 - 4x + 5)^{3.1} \cdot (4x - 4)$$
(b)
$$f'(x) = e^{x^4 - 3x + 1} \cdot (4x^3 - 3)$$
(c)
$$f'(x) = \frac{1}{4x^3 - 4x + 7} \cdot (12x^2 - 4)$$

Example 9.2. Compute the derivative of $f(x) = e^{(3x^2 - 3x + 9)^3}$. This requires the chain rule twice...but don't over-think this one.

$$\begin{aligned} \frac{df}{dx} &= e^{(3x^2-3x+9)^3} \cdot \frac{d}{dx}(3x^2 - 3x + 9)^3 \\ &= e^{(3x^2-3x+9)^3} \cdot 3(3x^2 - 3x + 9)^2 \cdot (6x - 3) \\ &= 9(2x - 1)(3x^2 - 3x + 9)^2 e^{(3x^2-3x+9)^3} \end{aligned}$$

Example 9.3. Using the chain rule, if $f(u) = u^3 - e^u + 1$ and $u = 2x + 1$, compute $\frac{df}{dx}$.

The chain rule states that $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.
**note: the u in this example is the $g(x)$ in the chain rule stated above.*

So, since $\frac{df}{du} = 3u^2 - e^u$ and $\frac{du}{dx} = 2$, we have that

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = (3u^2 - e^u)(2) = 2\Big(3(2x + 1)^2 - e^{2x+1}\Big)$$

Product and Quotient Rules

The product and quotient rules allow us to take the derivative of a **function multiplied/divided by another function**.

Product Rule

$$\boxed{\frac{d}{dx}\big[f(x) \cdot g(x)\big] = \frac{df}{dx} \cdot g(x) + f(x) \cdot \frac{dg}{dx}}$$

Simpler form: $(fg)' = f'g + fg'$.

Quotient Rule

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{df}{dx} \cdot g(x) - f(x) \cdot \frac{dg}{dx}}{\big(g(x)\big)^2}}$$

Simpler form: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Example 10.1. Compute the derivatives of the following functions.

$$\text{(a) } f(x) = (3x^2 - x + 1)^5 e^{2x-1}, \quad \text{(b) } f(x) = \frac{\ln x}{(x-1)^5}$$

(a) Use the product rule.

$$f'(x) = 5(3x^2-x+1)^4(6x-1)\cdot e^{2x-1} + (3x^2-x+1)^5 \cdot e^{2x-1}(2) = e^{2x-1}(3x^2-x+1)^4\Big(5(6x-1)+2(3x^2-x+1)\Big)$$

(b) Use quotient rule.

$$f'(x) = \frac{\frac{1}{x}\cdot(x-1)^5 - (\ln x)\cdot 5(x-1)^4}{\big((x-1)^5\big)^2} = \frac{\frac{1}{x}\cdot(x-1)^5 - (\ln x)\cdot 5(x-1)^4}{(x-1)^{10}} = \frac{\frac{x-1}{x} - 5\ln x}{(x-1)^6}$$

Instantaneous and Average Rates of Change

Instantaneous rate of change: The rate of change at a specific point $x = a$,

$$\boxed{f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}}$$

**note: in this class, we just use $f'(a)$, not the limit definition of the derivative.*

Average rate of change: from point $x = a$ to $x = b$ is defined as

$$\boxed{\frac{f(b) - f(a)}{b - a}}$$

Example 11.1. Compute the instantaneous rate of change of the function $f(t) = 4t^2 - 5t + 9$ at $t = 4$. Then, compute the average rate of change over the interval $[1, 3]$.

$$f'(4) = [8t - 5]_{t=4} = 27$$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{(4\cdot 3^2 - 5\cdot 3 + 9) - (4\cdot 1^2 - 5\cdot 1 + 9)}{3} = \frac{30 - 8}{3} = \frac{22}{3}$$

Example 11.2. The demand equation of a certain commodity x (units) is given by

$$p(x) = x^2 - 4x + 20.$$

What is the instantaneous rate of change of the revenue when $x = 3$ (i.e., what is the marginal revenue when $x = 3$)? Also, what is the revenue's average rate of change over the interval $[1, 2]$?
**note: see Appendix J for the economics equations.*

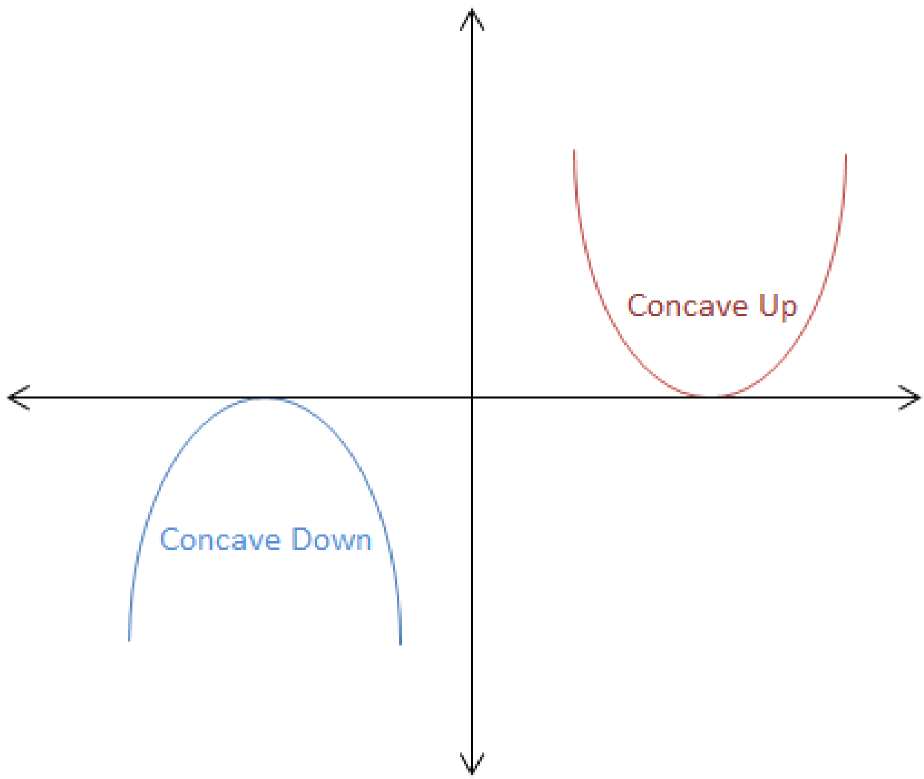
$$R'(3) = \frac{d}{dx}(xp(x))_{x=3} = \frac{d}{dx}(x^3 - 4x^2 + 20x)_{x=3} = (3x^3 - 8x + 20)_{x=3} = 81 - 24 + 20 = 77 \text{ dollars/unit}$$

$$\frac{R(2) - R(1)}{2 - 1} = \frac{(2^3 - 4\cdot 2^2 + 20\cdot 2) - (1^3 - 4\cdot 1^2 + 20\cdot 1)}{2 - 1} = 15$$

Interpreting the First and Second Derivatives

Recall that the **first derivative** of a function $f(x)$ indicates if $f(x)$ is *increasing* ($f'(x) > 0$) or *decreasing* ($f'(x) < 0$). It turns out the **second derivative** of a function $f(x)$ indicates the concavity/"shape" of $f(x)$, as seen in the figure below.

A function $f(x)$ is **concave up** if $f''(x) > 0$.
A function $f(x)$ is **concave down** if $f''(x) < 0$.



Below is a useful table to remember what each derivative represents.

	Positive (> 0)	Negative (< 0)
$f(x)$	Above $x-$axis	Below $x-$axis
$f'(x)$	Increasing ($\nearrow$)	Decreasing ($\searrow$)
$f''(x)$	Concave Up ($\cup$)	Concave Down ($\cap$)

There are a few more terms that we need to define.

Critical Point: a point $(a, f(a))$ where $f'(a) = 0$, i.e., the first derivative is zero.
Local/Relative Minimum: a point $(a, f(a))$ where $f'(a) = 0$ and $f(x)$ switches from *decreasing* to *increasing*, e.g., the red graph in the figure above.
Local/Relative Maximum: a point $(a, f(a))$ where $f'(a) = 0$ and $f(x)$ switches from *increasing* to *decreasing*, e.g., the blue graph in the figure above.
Absolute Minimum: the minimum value a function $f(x)$ attains on its domain.
**note: if $-\infty$, then DNE.*
Absolute Maximum: the maximum value a function $f(x)$ attains on its domain.
**note: if ∞, then DNE.*
Inflection Point: a point $(a, f(a))$ where $f''(a) = 0$ and $f(x)$ switch from *concave up* to *concave down* or vice versa.

There are two tests we can use to determine the local/relative minimum(s) and maximum(s).

First Derivative Test.

Step 1: Find all critical points, i.e., solve $f'(x) = 0$.
Step 2: Plug in other $x-$values to determine the intervals where $f(x)$ is increasing and decreasing. If decreasing to increasing, then relative minimum. If increasing to decreasing, then relative maximum.
**note: if increasing to increasing or decreasing to decreasing, then not a relative point.*
**hint: try using a number line for this step, as seen in the example below.*

Second Derivative Test.

Step 1: Find all critical points, i.e., solve $f'(x) = 0$.
Step 2: Plug critical points into $f''(x)$. If $f'' > 0$, then relative minimum. If $f'' < 0$, then relative maximum.
**note: if $f'' = 0$, then inconclusive by this test; try the first derivative test.*

Example 12.1. Find the point(s) at which the function $f(x) = 2x^3 - 3x^2 - 12x + 9$ attains its relative minimum and relative maximum. Use the first derivative test, and then use the second derivative test.

First Derivative Test:

Step 1: Solve $0 = f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$.
We have that the critical values are $x = 2$ and $x = -1$.
Step 2: Make a number line, as shown. The two critical values create three intervals: $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$. We want to know if $f(x)$ is *increasing* or *decreasing* in each of these intervals. So, we choose three "test points" in each interval to *plug into the derivative* $f'(x)$ and see if $f(x)$ is increasing or decreasing.

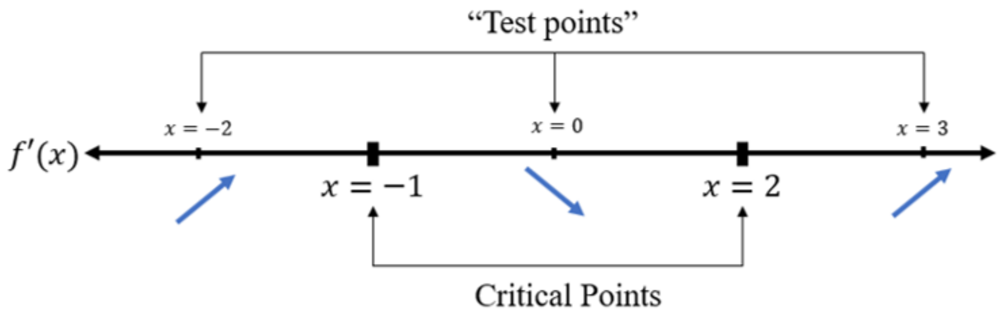
Since the test points can be *anything* in each interval, we clearly want to choose easy numbers.

Let's choose -2, 0, and 3.

$$f'(-2) = 6(-2-2)(-2+1) = 24 > 0 \quad \Rightarrow \quad \text{increasing}$$

$$f'(0) = 6(0-2)(0+1) = -12 < 0 \quad \Rightarrow \quad \text{decreasing}$$

$$f'(3) = 6(3-2)(3+1) = 24 > 0 \quad \Rightarrow \quad \text{increasing}$$



Thus, the relative minimum occurs when $x = 2$ and the relative maximum occurs when $x = -1$. Plugging these values into $f(x)$ to get the $y-$values, the points are

Relative Minimum: $(2, -11)$, Relative Maximum: $(-1, 16)$

Second Derivative Test:

Step 1: As before, the critical values are $x = -1$ and $x = 2$.
Step 2: $f''(x) = 12x - 6$.

$$f''(-1) = 12(-1) - 6 = -18 < 0 \quad \Rightarrow \quad \text{Relative Maximum}$$

$$f''(2) = 12(2) - 6 = 18 > 0 \quad \Rightarrow \quad \text{Relative Minimum}$$

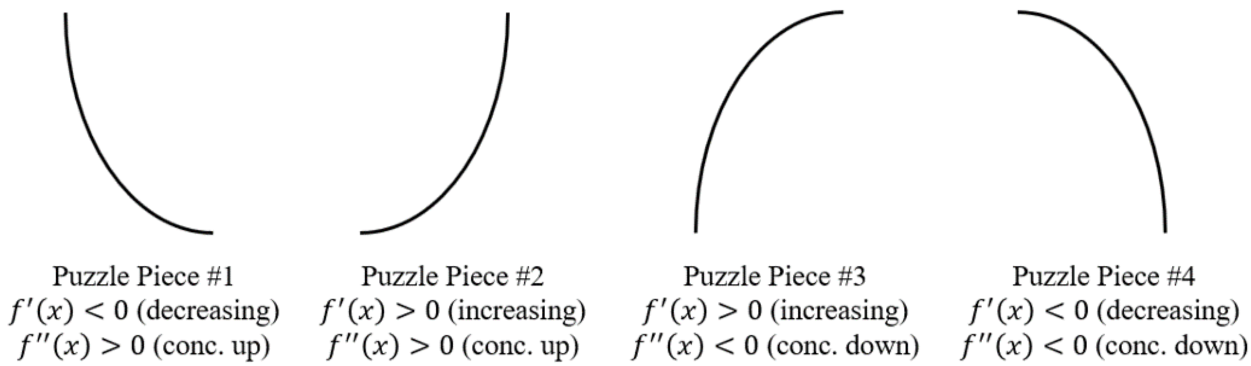
**note: $f''(-1) < 0$ means that the function is concave down at $x = -1$, i.e., $x = -1$ is the "top of a hill."*
**note: $f''(2) > 0$ means that the function is concave up at $x = 2$, i.e., $x = 2$ is the "bottom of a trough."*

Curve Sketching

We will use number lines for $f(x)$, $f'(x)$, and $f''(x)$ to sketch polynomials.

There are three steps to curve sketching:

Step 1: Find $x-$intercepts, critical points, and possible inflection points. In other words, solve $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.
Step 2: Create three number lines: $f(x)$, $f'(x)$, and $f''(x)$.
Step 3: Going from left to right, choose your "puzzle piece" from one of the four choices below.
Step 4: Put all puzzle pieces together.



Example 13.1. Sketch the same function as in the section "Interpreting the First and Second Derivatives," $f(x) = 2x^3 - 3x^2 - 12x + 9$, given that the $x-$intercepts are $x = 3$, $x = \dfrac{-3 \pm \sqrt{33}}{4} \approx 0.69, -2.19$.

Step 1: As shown in example 12.1,

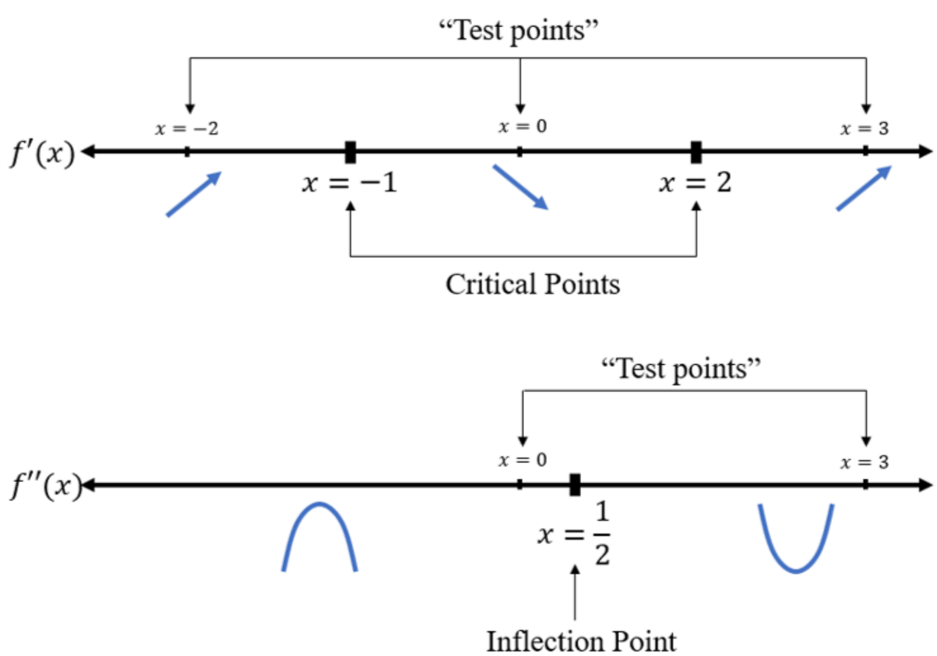
$$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1), \qquad f''(x) = 12x - 6 = 6(2x-1)$$

The number line for $f'(x)$ was done in example 12.1. The number line for $f''(x)$ is done similarly. Choosing $x = 0$ and $x = 3$ as our "test points" for the second derivative,

$$f''(0) = -6 < 0 \quad \Rightarrow \quad \text{Concave Down}$$

$$f''(3) = 30 > 0 \quad \Rightarrow \quad \text{Concave Up}$$

Step 2:

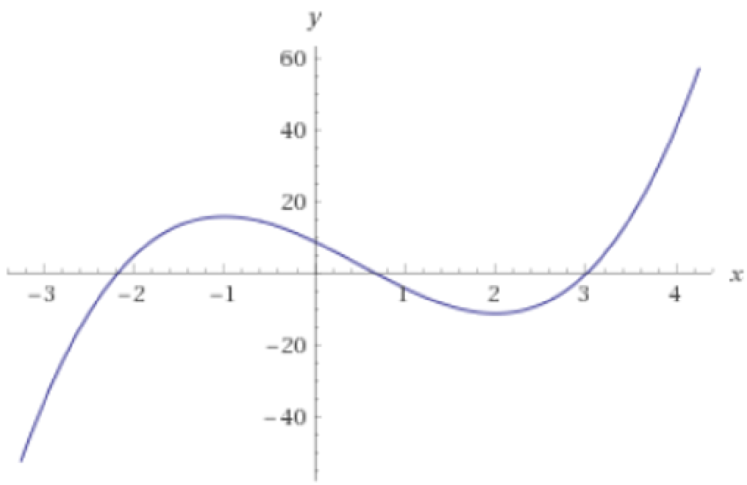


Step 3: There are *three* important values (i.e., critical values and inflection points). They make *four* important intervals, $(-\infty, -1)$, $(-1, 0.5)$, $(0.5, 2)$, and $(2, \infty)$. Looking at the number lines, we choose the correct puzzle piece for each interval.

$(-\infty, -1)$ gets puzzle piece # 3. $(-1, 0.5)$ gets puzzle piece # 4.

$(0.5, 2)$ gets puzzle piece # 1. $(2, \infty)$ gets puzzle piece # 2.

Step 4: Now, we put all these puzzle pieces together, making sure that we satisfy the $x-$intercepts.



Optimization

What types of problems? Optimization problems ask to *maximize* or *minimize* some variable. For instance, you might be asked to minimize the cost of building a boat; maximize the volume of a box under certain conditions; maximize the amount of light that shines through a window.

There are three general steps to optimization problems.

Step 1: Get the *objective* and *constraint* equations.
- **objective equation:** what you are trying to maximize/minimize.
- **constraint equation(s):** whatever other condition(s) must hold.
**note: you only need constraint equations if your objective equation has more than one independent variable!*
Step 2: Plug constraint equation(s) into objective equation.
Step 3: Find relative maximum/minimum.

Example 14.1. We want to ship a cylindrical package in the mail. However, the post office only allows us to ship cylindrical packages in which the length plus the girth is 85 inches. Find the *dimensions* that maximize the volume of the cylindrical package.

Step 1: Find the equations.
We want to *maximize the volume*. Thus, our **objective equation** is the *volume equation*. We know that the volume of a cylinder is

$$V = \pi r^2 l,$$

where r is the radius of the cylinder, and l is the length of the cylinder.

Since the objective equation has two independent variables, we need a **constraint equation** because we can't take the derivative unless we only have one independent variable (either r or l...it doesn't matter which one). We were told that the girth plus the length of the cylindrical package equals 85. Girth is just the circumference ($2\pi r$). So, our **constraint equation** is

$$2\pi r + l = 85.$$

Step 2: Plug constraint equation into objective equation.
We first rewrite the constraint equation

$$2\pi r + l = 85 \Rightarrow l = 85 - 2\pi r.$$

Now we can plug the constraint equation into the objective equation,

$$V = \pi r^2 l = \pi r^2(85 - 2\pi r) = 85\pi r^2 - 2\pi^2 r^3.$$

Step 3: Solve $V' = 0$ and find relative maximum.
Solving $V' = 0$ to find the critical points,

$$V'(r) = 170\pi r - 6\pi^2 r^2$$

$$\begin{aligned} 0 &= 170\pi r - 6\pi^2 r^2 \\ &= r(170\pi - 6\pi^2 r) \\ &\Rightarrow r = 0 \quad 170\pi r - 6\pi^2 r^2 = 0 \\ &\Rightarrow r = 0 \quad r = \frac{170}{6\pi} \end{aligned}$$

We just need to know which critical point is the relative maximum using either the *first or second derivative tests*; here we will use the second derivative test.

$$V''(r) = 170\pi - 12\pi^2 r \quad \Rightarrow \quad V''\left(\frac{170}{6\pi}\right) = 170\pi - 340\pi < 0 \quad \Rightarrow \quad \text{Relative Maximum}$$

Thus, $r = \dfrac{170}{6\pi} = 9.02$ inches is the relative maximum.

We found one dimension, r. To find the other dimension, simple plug $r = \dfrac{170}{6\pi}$ into the constraint equation to find the length, l.

$$l = 85 - 2\pi r = 85 - 2\pi\frac{170}{6\pi} = 85 - \frac{170}{3} = 28.33 \text{ inches.}$$

So, the dimensions that maximize the volume are $r = 9.02$ inches, and $l = 28.33$ inches.

Example 14.2. Find the point on the line $y = -2x + 5$ that is closest to the origin.

Step 1: Find the equations.
The Pythagorean theorem tells us that

$$z^2 = x^2 + y^2,$$

where z is the hypotenuse (i.e., the distance to the origin). Thus, we want to *minimize* z. Well, we were already given the constraint equation.

Step 2: Plug constraint equation into objective equation.
Plugging the constraint equation into the main/objective equation,

$$z^2 = x^2 + (-2x + 5)^2.$$

Step 3: Solve $z' = 0$ and find relative minimum.
Differentiating each side with respect to x,

$$\begin{aligned} 2z\frac{dz}{dx} &= 2x + 2(-2x+5)(-2) \\ \frac{dz}{dx} &= \frac{10x - 20}{2z} \\ &= \frac{10x-20}{2\sqrt{x^2 + (-2x+5)^2}} \\ &= \frac{5x-10}{\sqrt{x^2 + (-2x+5)^2}} \end{aligned}$$

Finding all relative points,

$$0 = \frac{5x-10}{\sqrt{x^2 + (-2x+5)^2}} \quad \Rightarrow \quad x = 2.$$

One can verify with the first or second derivative tests that $x = 2$ is the relative minimum. And so, $y = -2(2) + 5 = 1$. The point is $(2, 1)$.

Example 14.3. A rectangular storage container with an open top and square base is to have a volume of 10 cubic meters. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

Step 1: Find the equations.
We want to minimize the cost, so the main/objective equation is a cost function,

$$C = 10(\underbrace{y^2}_{btm.}) + 6(\underbrace{4xy}_{sides}) = 10y^2 + 24xy,$$

where y is the dimension of the base and x is the height. The constraint equation is

$$10 = xy^2 \quad \Rightarrow \quad x = \frac{10}{y^2}$$

Step 2: Plug constraint equation into objective equation.
Plugging the constraint equation into the objective equation,

$$C = 10y^2 + 24 \cdot \frac{10}{y^2} \cdot y = 10y^2 + \frac{240}{y}.$$

Step 3: Solve $C' = 0$ and find relative minimum.
Finding the relative point(s),

$$0 = \frac{dC}{dy} = 20y - \frac{240}{y^2} \quad \Rightarrow \quad 0 = 20y^3 - 240 \quad \Rightarrow \quad y \approx 2.29 \text{ m}.$$

We can use the first or second derivative test to easily verify this is the minimum. So, the cost of materials for the cheapest such container is

$$C = 10(2.29)^2 + \frac{240}{2.29} \approx 157.2 \text{ dollars}.$$

Implicit Differentiation

Up until this point, we've only had to deal with finding derivatives of functions given to us in the form $y =$ "stuff with x". However, functions are not always given in such a nice equation. What if we were asked to find $\frac{dy}{dx}$ for the function $e^y(y + x) = 2x + 1$? Notice that it is not possible -or at least not easy enough- to rewrite this equation in the form $y =$ "only stuff with x." What is one to do in this situation? You just take the appropriate derivative as usual, being very careful with the chain rule.

Whenever you take a derivative with respect to the *independent variable (e.g., x, t)* of a *dependent variable (e.g., y)*, you must also multiply by y' because of the chain rule. Please see the section on the chain rule if you are still not comfortable with this.

Here are the major steps to implicit differentiation:

Step 1. Take the appropriate derivative of both sides of the equation.
Step 2. Factor out y' (i.e., we want to get y' all by itself).
Step 3. Divide as necessary to get y' all by itself.

Example 15.1. Find $\frac{dy}{dx}$, given the equation $y(y - x) = 4x^3$.

$$\begin{aligned} \frac{d}{dx} y(y-x) &= \frac{d}{dx} 4x^3 \\ \underbrace{(1)y'(y-x) + y((1)y' - 1)}_{\text{product rule}} &= 12x^2 \\ y'(y - x + y) &= 12x^2 + y \\ y' &= \frac{12x^2 + y}{2y - x}. \end{aligned}$$

Observe that our solution, $\dfrac{dy}{dx} = \dfrac{12x^2 + y}{2y - x}$ has both y's and x's. THIS IS OKAY!!!

All I asked was to find $\dfrac{dy}{dx}$; I didn't specify that it only had to have x's.

Example 15.2. Find $\dfrac{dy}{dx}$, given the equation $3x^3y^4 = \dfrac{2x-1}{2y}$.

$$\begin{aligned}
\frac{d}{dx}3x^3y^4 &= \frac{d}{dx}\frac{2x-1}{2y} \\
3(3x^2y^4 + x^3 4y^3 y') &= \frac{(2)2y - (2x-1)2y'}{(2y)^2} \\
y'(12x^3y^3) + 9x^2y^4 &= \frac{4y - y'(4x-2)}{4y^2} \\
y'(12x^3y^3) + \frac{y'(4x-2)}{4y^2} &= \frac{4y}{4y^2} - 9x^2y^4 \\
y'\Big(12x^3y^3 + \frac{2x-1}{2y^2}\Big) &= \frac{1}{y} - 9x^2y^4 \\
y' &= \frac{\frac{1}{y} - 9x^2y^4}{12x^3y^3 + \frac{2x-1}{2y^2}}.
\end{aligned}$$

Example 15.3. Find the tangent line equation for $y = f(x)$ at the point where $x = 2$, given the equation $y^2 + x^4 = (2x-1)^4$.

The equation for the desired tangent line is $y - f(2) = f'(2)(x-2)$. Thus, we need to know the y-value when $x = 2$. Plugging in $x = 2$, we get

$$(f(2))^2 + 2^4 = (2(2)-1)^4 \Rightarrow (f(2))^2 + 16 = 81 \Rightarrow f(2) = \sqrt{65}.$$

Next, we need to know the slope $f'(2)$, i.e., we need to know $\dfrac{dy}{dx}|_{x=2}$. Obviously, we first need to find $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}y^2 + x^4 &= \frac{d}{dx}(2x-1)^4 \\
2yy' + 4x^3 &= 4(2x-1)^3(2) \\
2yy' &= 8(2x-1)^3 - 4x^3 \\
y' &= \frac{4(2x-1)^3 - 2x^3)}{y}.
\end{aligned}$$

Therefore, plugging in $(x_1, y_1) = (2, \sqrt{65})$, we get that $m = \dfrac{4(2(2)-1)^3 - 2(2)^3}{\sqrt{65}} = \dfrac{92}{\sqrt{65}}$.

Thus, the tangent line equation is $y - \sqrt{65} = \dfrac{92}{\sqrt{65}}(x-2)$.

Example 15.4. Find $\frac{dy}{dt}$, given the equation $2x + 4y^4 = x^4y^2$ and the fact that both x and y are functions of time.
Please note that unlike before, where y was a function of x, now BOTH are functions of time, t.

$$\begin{aligned} \frac{d}{dt}(2x + 4y^4) &= \frac{d}{dt}(x^4y^2) \\ 2x' + 16y^3y' &= (4x^3x')(y^2) + (x^4)(2yy') \\ y'(16y^3 - 2x^4y) &= 4x^3y^2x' - 2x' \\ \frac{dy}{dt} &= \frac{4x^3y^2\frac{dx}{dt} - 2\frac{dx}{dt}}{16y^3 - 2x^4y}. \end{aligned}$$

Related Rates

The goal is to find the *rate of change* of some variable; for the sake of this course, the rate of change will always be with respect to time (i.e., $\frac{d}{dt}$).

Here are the three general steps for solving related rates problems:

Step 1: Figure out the required equation.
Step 2: Take a derivative with respect to time, $\frac{d}{dt}$, being careful when using implicit differentiation/chain rule.
Step 3: Plug in all other information provided.

Example 16.1. A cylindrical tank with radius 5m is being filled with water at a rate of 3 cubic meters per minute. How fast is the height of the water increasing? Include units.

Step 1: The equation for the volume of a cylinder is

$$V = \pi r^2 h,$$

where V is the volume, r is the radius, and h is the height.

Step 2: Taking a derivative with respect to time,

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

**note: r is constant and not dependent on time, i.e., radius is not changing.*

Step 3: We want to find $\frac{dh}{dt}$ at the time when $r = 5$ and $\frac{dV}{dt} = 3$. Plugging in these values,

$$3 = \pi(5)^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{3}{25\pi} \text{ meters per minute.}$$

Example 16.2. If $x^2 + y^2 + z^2 = 9$, $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 4$, then find $\frac{dz}{dt}$ when $(x, y, z) = (2, 2, 1)$.

Taking a derivative with respect to time,

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} + 2z \cdot \frac{dz}{dt} = 0.$$

Plugging in the known values,

$$2(2)(5) + 2(2)(4) + 2(1)\frac{dz}{dt} = 0 \quad \Rightarrow \quad \frac{dz}{dt} = -\frac{36}{2} = -18.$$

Example 16.3. The length of a rectangle is increasing at a rate of 8cm/s and its width is increasing at a rate of 3cm/s. When the length is 20cm and the width is 10cm, how fast is the area of the rectangle increasing? Include units.

Step 1: We want to know $\frac{dA}{dt}$. Our equation is

$$A = xy,$$

where x is length and y is width.

Step 2: Taking a derivative with respect to time,

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}.$$

Step 3: Plugging in known values,

$$\frac{dA}{dt} = (8)(10) + (20)(3) = 140 \text{ cm}^2/\text{s}.$$

Example 16.4. A ladder of length 5m is leaning against a vertical wall. The base of the ladder is then pulled away from the wall at a rate of 0.5m/s. At the moment at which the base of the ladder is 3m from the wall, how fast is the top of the ladder sliding down the wall?

Step 1: Let x be the distance from the *base* of the ladder and y be the distance of the *top* of the ladder from the ground. We know

$$x^2 + y^2 = 25,$$

and we want to find $\frac{dy}{dt}$ at the time when $x = 3$, $y = 4$ (which you find from the equation), and $\frac{dx}{dt} = +0.5$.

Steps 2 and 3: Differentiating with respect to time,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{3}{8}.$$

So, the top of the ladder is sliding *down* the wall at a rate of $\frac{3}{8}$m/s.

Logarithmic Differentiation

The product and quotient rules are ways to differentiate two functions either being multiplied and/or divided. However, what if there are more than three functions being multiplied and/or divided? Surely, you would have to be absolutely crazy to do the product or quotient rule over and over again... In these situations, we use **logarithmic differentiation**.

Idea: Use the properties of logarithms (see Appendix D) to "split up" the complicated function so that it becomes easier to work with.
When to use logarithmic differentiation: There are two types of problem for which we will use logarithmic differentiation.
1. When several functions are being multiplied/divided.
2. When you have a function of the form $[\text{stuff}]^{\text{stuff with } x}$.

There are three steps to logarithmic differentiation:
Step 1: Take ln of each side (unless ln is already in the given equation).
Step 2: Use as many properties of ln as possible!
Step 3: Take derivative.

Example 17.1. Using *logarithmic differentiation*, find $\dfrac{df}{dx}$ for the following:

$$\text{(a) } f(x) = \sqrt[x]{3}, \qquad \text{(b) } f(x) = \frac{e^{3x^3+4x}(2x+7)}{(x-1)^3}$$

(a)

$$\begin{aligned} \ln f &= \ln \sqrt[x]{3} \\ &= \ln 3^{1/x} \\ &= \frac{1}{x} \ln 3 \end{aligned}$$

Taking a derivative of each side,

$$\begin{aligned} \frac{f'}{f} &= (\ln 3) \cdot \frac{-1}{x^2} \\ \text{So } f'(x) &= f(x) \cdot \frac{-\ln 3}{x^2} \\ f'(x) &= \frac{-(\ln 3) \sqrt[x]{3}}{x^2} \end{aligned}$$

(b)

$$\begin{aligned} \ln f &= \ln \frac{e^{3x^3+4x}(2x+7)}{(x-1)^3} \\ &= \ln e^{3x^3+4x} + \ln (2x+7) - \ln (x-1)^3 \\ &= 3x^3 + 4x + \ln (2x+7) - 3 \ln (x-1) \end{aligned}$$

Taking a derivative of each side,

$$\frac{f'}{f} = 9x^2 + 4 + \frac{2}{2x+7} - \frac{3}{x-1}$$

$$\text{So } f'(x) = f(x) \cdot \left(9x^2 + 4 + \frac{2}{2x+7} - \frac{3}{x-1}\right)$$

$$f'(x) = \frac{e^{3x^3+4x}(2x+7)}{(x-1)^3} \cdot \left(9x^2 + 4 + \frac{2}{2x+7} - \frac{3}{x-1}\right)$$

Example 17.2. Find the derivative of $f(x) = \ln{(x-1)(x+4)}$. Note that we didn't have to do step 1 since $f(x)$ was already given in ln. So, we skip to step 2.

$$f(x) = \ln{(x-1)(x+4)} = \ln{(x-1)} + \ln{(x+4)}.$$

Now, when we take a derivative of each side, we DO NOT need to use the chain rule on the left side since it is $f(x)$, not $\ln{(f(x))}$.

$$f'(x) = \frac{1}{x-1} + \frac{1}{x+4}.$$

Exponential Growth and Decay

The solution to the differential equation

$$\boxed{\frac{dy}{dt} = ky, \qquad y(0) = y_0}$$

is

$$\boxed{y(t) = y_0 e^{kt}}$$

where k is the growth/decay constant, $y(t)$ is the dependent variable, $y'(t)$ is the rate of change of the dependent variable, and y_0 is the initial value.

Qualitative analysis: We can tell a lot about $y(t)$ just from the differential equation and its solution. For instance, k is the **growth/decay constant**. Thus, it should at least be intuitively obvious that
1. If $k < 0$ (negative), then the solution is *decaying.*
2. If $k > 0$ (positive), then the solution is *growing.*
Whenever you are told when a population grows/dies (i.e., doubles, quadruples, half-life, ...), we typically use this information to determine the growth/decay constant k. Next is an example of this type of problem.

Example 18.1. Let $P(t)$ be the population of cows (in thousands), t years after 2010. If the population doubled from 2010 to 2013, then what is the growth constant k?

We know that the solution to this problem has the form $P(t) = P_0 e^{kt}$. Note that we do not necessarily know what P_0 is, so we let it be arbitrary. If the population doubled at $t = 3$ years, then plugging in $t = 3$,

$$\underbrace{2P_0}_{\text{twice population in 2010}} = P_0 e^{3k}.$$

Solving by dividing each side by P_0 (this is why it was okay we didn't know P_0),

$$2 = e^{3k} \quad \Rightarrow \quad k = \frac{\ln 2}{3}.$$

Thus, the solution is

$$P(t) = P_0 e^{\frac{\ln 2}{3} t}.$$

Here is another type of problem. If given the differential equation, we can easily find the solution. However, we could also be given the solution and be expected to find the differential equation.

Example 18.2. The amount of some element (in milligrams) $y(t)$ after t years is modeled by the solution

$$y(t) = 1000e^{-0.2t}.$$

What was the differential equation?

Clearly, the initial amount was 1000 milligrams. Moreover, referring to the general solution, it is also clear that $k = -0.2$ (which indicates decay). Thus, the differential equation $y' = ky$, $y(0) = y_0$ is

$$\frac{dy}{dt} = -0.2y, \quad y(0) = 1000.$$

Now, what is the half-life of the element? In other words, when was the amount of the element 500 milligrams? Solving

$$500 = 1000e^{-0.2t} \quad \Rightarrow \quad \frac{1}{2} = e^{-0.2t} \quad \Rightarrow \quad t = -5\ln(0.5) \approx 3.47 \text{ years}.$$

Example 18.3. Given the differential equation $P'(t) = 4P(t)$, $P(0) = 1500$, which models the number of sheep $P(t)$ after t years, answer the following:
(a) At what rate is the population increasing when there are 5000 sheep?
We just have to use the differential equation $P'(t) = 4P(t)$. We know the population P, and want the rate of change $P'(t)$. Thus, when there are 5000 sheep, the population is increasing at a rate of $4 \cdot 5000 = 20000$ sheep/year.

(b) How many sheep are there at the time when the population is increasing at a rate of 10000 sheep/year?
Similar to part (a), we know $P'(t)$ and want the population $P(t)$. Thus, when the population of sheep is growing at a rate of 10000 sheep/year, the population is $\frac{10000}{4} = 2500$ sheep.

Compound Interest

Suppose we accumulate interest on a principle P (in dollars); P is the "starting amount" and is also called the present value. The interest rate could be compounded several times a year, e.g., monthly, weekly, daily. In real-world applications such as interest on a loan or the amount of money in a savings account, we really only care about knowing how much money has accumulated in terms of years. Before giving you the equations, let's write some terminology/notation.

t = time in **whole years** (i.e., no fractions)

T = time in **whole periods** (e.g., months, weeks, days)

m = number of periods in a year (e.g., 12 months per year, 52 weeks per year)

P = the principle amount in dollars

A = the accumulated amount in dollars

I = the interest accumulated in dollars

r = nominal interest rate per year

i = interest rate per conversion period (e.g., interest rate per month)

An important relationship - the *interest rate per conversion period* is just the *nominal interest rate per year* divided by the number of periods in a year.

$$i = \frac{r}{m} = \frac{\text{annual interest rate}}{\text{periods per year}}$$

(Discrete) Compound Interest Formula. We call this the "discrete" compound interest formula because in this case we assume interest is accumulated periodically/discretely, e.g., interest is compounded every month. Assume that interest at a nominal rate r per year is compounded m times per year on a principle amount P.

$$A(t \text{ years}) = P\Big(1 + \frac{r}{m}\Big)^{mt} \qquad \text{or} \qquad A(T \text{ periods}) = P(1+i)^T$$

Deriving the (discrete) compound interest formula.

As we know from economics and finance, the interest after T periods is

$$I = (\text{principle amount}) \times (\text{interest rate per conversion period}) \times (\text{periods}) = PiT$$

And so we know that the accumulated amount after T periods is

$$A = (\text{principle amount}) + (\text{interest}) = P + I = P + PiT = P(1 + iT)$$

If we start with principle amount P, then **after 1 period** the accumulated amount is $P(1+i(1))$ dollars. Now, the "new principle/starting amount" is $P(1 + i(1))$. And so, the accumulated amount **after another period (i.e., 2 periods in total)** is $P(1+i(1))(1+i(1))$ dollars. Now, the "new principle/starting amount" is $P(1 + i)(1 + i) = P(1 + i)^2$. We can continue this for **T periods in total** so that

$$A(T) = P(1+i)^T$$

Converting time back in terms of whole years $T = mt$,

$$A(t) = P\Big(1 + \frac{r}{m}\Big)^{mt}$$

Example 19.1. How long will it take for \$1000 in a savings account to grow to \$8000 if it earns interest at a (nominal) rate of 4% per year compounded monthly?

We know $P = 1000$, $A = 8000$, $m = 12$, and $r = 0.04$.

$$8000 = 1000\left(1 + \frac{0.04}{12}\right)^{12t} \quad \Rightarrow \quad \ln(8) = 12t\ln\left(1 + \frac{0.04}{12}\right) \quad \Rightarrow \quad t \approx 52 \text{ years.}$$

Example 19.2. We have $P = 2000$ dollars in a savings account. Find the accumulated amount after 5 years if the nominal interest rate of 5% per year is compounded: (a) annually, (b) quarterly, or (c) monthly.

(a) $A(t=5) = 2000\left(1 + \frac{0.05}{1}\right)^{(1)(5)} \approx 2552.56$ dollars

(b) $A(t=5) = 2000\left(1 + \frac{0.05}{4}\right)^{(4)(5)} \approx 2564.07$ dollars

(c) $A(t=5) = 2000\left(1 + \frac{0.05}{12}\right)^{(12)(5)} \approx 2566.72$ dollars

A VERY IMPORTANT OBSERVATION - Notice in the previous example that the interest earned depends on the frequency with which the interest is compounded. However, the nominal interest rate does not take into account the compounding period. We saw in the previous example that compounding interest more frequently produced greater interest. Although this is nice when dealing with a savings account, this would be far from ideal if dealing with a loan!

Issue: The nominal interest rate does not take into account the compounding period and hence can be misleading.
Fix: We need a way to compare interest rates on an "even playing field," where the compounding period is accounted for. The way we do this is by the **effective interest rate (a.k.a., annual percentage yield (APY))** r_{eff}.

The effective rate is the simple annual interest rate that would produce the same accumulated amount in 1 year as the nominal rate compounded m times a year. In other words, it can be considered the annual interest rate that accounts for the compounding period.

$$\boxed{r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1}$$

The effective interest rate is easily derived by rearranging the (discrete) compound interest formula so that the "period" is 1.

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt} = P\left(1 + \left(1 + \frac{r}{m}\right)^m - 1\right)^t = P(1 + r_{\text{eff}})^t = P\left(1 + \frac{r_{\text{eff}}}{1}\right)^{(1)(t)}$$

Fun fact: The Truth in Lending Act of 1968 passed by Congress requires that the effective interest rate be disclosed in all contracts involving interest charges.

Example 19.3. Compute the effective interest rates for the previous example.

(a) $r_{\text{eff}} = \left(1 + \dfrac{0.05}{1}\right)^1 - 1 = 5.00\%$

(b) $r_{\text{eff}} = \left(1 + \dfrac{0.05}{4}\right)^4 - 1 \approx 5.09\%$

(c) $r_{\text{eff}} = \left(1 + \dfrac{0.05}{12}\right)^{12} - 1 \approx 5.12\%$

Issue: We have only considered cases where interest is compounded periodically/discretely. What if interest is compounded continuously? In other words, instead of interest being compounded, say every month, it's compounded constantly!
Fix: This is pretty much asking - what happens as the number of periods per year is infinite? If interest is compounded constantly, then that's pretty much the same as a period of about zero, which means MANY MANY periods per year.

Continuous Compound Interest Formula

$$A(t) = Pe^{rt},$$

where r is the annual interest rate compounded continuously and t is in years.

The continuous compound interest formula is easily computed by taking the limit.

$$\lim_{m\to\infty} P\left(1 + \frac{r}{m}\right)^{mt} = \lim_{m\to\infty} P\left[\left(1 + \frac{r}{m}\right)^m\right]^t = P(e^r)^t = Pe^{rt}$$

Remark: We've observed that as m gets larger, so does the effective interest rate. It turns out that as $m \to \infty$, the effective interest rate approaches the largest value possible.

$$\lim_{m\to\infty} r_{\text{eff}} = \lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1$$

Example 19.4. The market value of a local investment company's property is given by $V(t) = 150,000e^{\sqrt{t/2}}$ dollars, where t is the time in years from the present. The expected rate of appreciation is 8% compounded continuously for the next 20 years. Find an expression for $P(t)$.

We want to use the continuous compound interest formula with $A(t) = V(t)$, $r = 0.08$, and $0 < t < 20$.

$$P(t) = A(t)e^{-rt} = 150,000e^{\sqrt{t/2}}e^{-0.08t} = 150,000e^{-0.08t+\sqrt{t/2}}, \qquad 0 < t < 20.$$

Indefinite Integration (a.k.a. Antidifferentiation)

What is the antiderivative/indefinite integral of a function $f(x)$**?** It is a function $F(x)$ such that $F'(x) = f(x)$. Essentially, integration "undoes differentiation." For instance, if the derivative of $3x^2 + C$ (where C is any constant) is $6x$, then the (indefinite) integral of $6x$ is $3x^2 + C$.
Notation: We denote the (indefinite) integral of a function $f(x)$ as

$$F(x) = \int f(x)dx$$

**note: the $\int$ symbol and the dx must both be written when taking the integral of a function, and the dx must also go at the end of the expression.*

Moreover, just like how $\frac{d}{dx}$ means "differentiate with respect to x" (see Appendix F on Leibniz notation), $\int f(x)dx$ means "integrate f with respect to x."

The 3 "Big Ones" of Integration

For the sake of this course, we only need to know the antiderivatives of three functions.

Polynomials, $f(x) = x^n$ **for** $n \neq -1$

$$\int f(x)dx = \frac{1}{n+1}x^{n+1} + C$$

Exponentials, $f(x) = e^{kx}$ **for** $k \neq 0$

$$\int f(x)dx = \frac{1}{k}e^{kx} + C$$

Natural Logarithm, $f(x) = \frac{1}{x}$

$$\int f(x)dx = \ln|x| + C$$

**note: Notice the $+C$ with each antiderivative. We need the $+C$ because the derivative of any constant C is zero.*

Moreover, the following two rules apply:

1. You can "add" antiderivatives: $\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx$

2. You can "scale" antiderivatives: $\int kf(x)dx = k\int f(x)dx$

Example 21.1. Find the antiderivatives of the following functions:

$$\text{(a) } f(x) = x^{8.1} + 4x^3, \quad \text{(b) } f(x) = 3e^{5x}, \quad \text{(c) } f(x) = 4x^{-1}$$

(a) The two laws tell us that we can simply add the antiderivatives of $x^{8.1}$ and $4x^3$.

$$\int (x^{8.1} + 4x^3)dx = \int x^{8.1}dx + 4\int x^3dx = \frac{1}{8.1+1}x^{8.1+1} + 4\cdot\frac{1}{3+1}x^{3+1} + C = \frac{x^{9.1}}{9.1} + x^4 + C$$

(b)

$$\int 3e^{5x}dx = 3\int e^{5x}dx = 3\cdot\frac{1}{5}e^{5x} + C = \frac{3}{5}e^{5x} + C$$

(c)

$$\int 4x^{-1}dx = 4\int \frac{1}{x}dx = 4\ln|x| + C$$

Finding C

If we know a point that $F(x) = \int f(x)dx$ goes through, then the constant C has to be a specific value.

Example 22.1. Find the function $f(x)$ such that $f'(x) = \frac{1}{x}$ and $f(e) = 2$.
Since integration "undoes" differentiation,

$$f(x) = \int f'(x)dx = \int \frac{1}{x}dx = \ln|x| + C$$

Now, we know that $f(e) = 2$,

$$2 = \ln|e| + C \quad \Rightarrow \quad C = 1.$$

So, $f(x) = \ln|x| + 1$.

Example 22.2. If $f(x)$ is the function such that $f'(x) = 2x$ and $f(-1) = 2$, then what is $f(2)$?

$$f(x) = \int f'(x)dx = \int 2xdx = x^2 + C$$

Now, we know that $f(-1) = 2$,

$$2 = (-1)^2 + C \quad \Rightarrow \quad C = 1$$

So,

$$f(x) = x^2 + 1 \quad \Rightarrow \quad f(2) = 5.$$

$u-$Substitution

We already know how to integrate three functions: x^n $(n \neq -1)$, e^{kx} $(k \neq 0)$, and $\frac{1}{x}$.
However, what if we don't have a function of one of three forms? What if we have to integrate a function comprised of several functions being multiplied or divided? One method to integrate such functions is $u-$**substitution** (or sometimes simply called "substitution").

Idea: Rewrite the harder function so that it becomes one of the three easier functions in terms of a new variable u,

$$\text{i.e.,} \quad u^n \ (n \neq -1), \quad e^{ku} \ (k \neq 0), \quad \frac{1}{u}$$

Here are the four general steps to integrating by $u-$substitution.

Step 1: Choose a u, also getting du.
Step 2: Substitute *all the x's* in the original function with u and du so that we get one of the three easier functions in terms of u.
**note: we should not have anymore x's in the function.*
Step 3: Integrate with respect to u.
Step 4: Write final answer in terms of x by plugging in u.

Example 23.1. Evaluate the following integrals by $u-$substitution.

$$\text{(a)} \int 3xe^{x^2-1}dx, \quad \text{(b)} \int \frac{(\ln x)^3}{x}dx, \quad \text{(c)} \int (2x-1)^8 dx$$

(a)
Step 1: First, look at $3xe^{x^2-1}$ and ask yourself, "which of the three *big ones* does it look like I want?" Well, it looks like we could have e^u if we let $u = x^2 - 1$.

$$u = x^2 - 1, \qquad du = 2xdx \leftrightarrow \frac{du}{2} = xdx$$

Step 2: Using u and du, we want to get rid of all the x's (including dx).

$$\int 3xe^{x^2-1}dx = 3\int e^{x^2-1}xdx = 3\int e^u \frac{du}{2} = \frac{3}{2}\int e^u du$$

Step 3: Integrate with respect to u.

$$\int \frac{3}{2}e^u du = \frac{3}{2}e^u + C$$

Step 4: Plug in u.

$$\frac{3}{2}e^u + C = \frac{3}{2}e^{x^2-1} + C$$

(b)
Step 1: First, look at $\frac{(\ln x)^3}{x}$ and ask yourself, "which of the three *big ones* does it look like I want?" Well, it looks like we could have u^3 if we let $u = \ln x$.

$$u = \ln x, \qquad du = \frac{1}{x}dx$$

Step 2: Using u and du, we want to get rid of all the x's (including dx).

$$\int \frac{(\ln x)^3}{x}dx = \int (\ln x)^3 \frac{dx}{x} = \int u^3 du$$

Step 3: Integrate with respect to u.

$$\int u^3 du = \frac{u^4}{4} + C$$

Step 4: Plug in u.

$$\frac{u^4}{4} + C = \frac{(\ln x)^4}{4} + C$$

(c)
Step 1: First, look at $(2x-1)^8$ and ask yourself, "which of the three *big ones* does it look like I want?" Well, it looks like we could have u^8 if we let $u = 2x - 1$.

$$u = 2x - 1, \qquad du = 2dx \leftrightarrow \frac{du}{2} = dx$$

Step 2: Using u and du, we want to get rid of all the x's (including dx).

$$\int (2x-1)^8 dx = \int u^8 \frac{du}{2} = \frac{1}{2}\int u^8 du$$

Step 3: Integrate with respect to u.

$$\frac{1}{2}\int u^8 du = \frac{1}{2}\cdot\frac{u^9}{9} + C = \frac{u^9}{18} + C$$

Step 4: Plug in u.

$$\frac{u^9}{18} + C = \frac{(2x-1)^9}{18} + C$$

Example 23.2. Here is a particular problem that causes most students difficulty when first encountered. Evaluate

$$\int x(x+2)^7 dx$$

Let $u = x+2$ and $du = dx$. However, plugging in u and du, we get

$$\int x(x+2)^7 dx = \int xu^7 du.$$

Notice that we still have an x left over! However, just because we already used $u = x+2$ doesn't mean we can't use it again. Can we still get rid of this last x using $u = x+2$? Yes!

$$u = x+2 \leftrightarrow x = u-2$$

So,

$$\int x(x+2)^7 dx = \int (u-2)u^7 du = \int (u^8 - 2u^7) du = \frac{u^9}{9} - \frac{2u^8}{8} + C = \frac{(x+2)^9}{9} - \frac{(x+2)^8}{4} + C$$

Example 23.3. Determine the integral using substitution.

$$\int \frac{e^x}{2+2e^x} dx$$

Let $u = 2+2e^x$. Then $du = 2e^x dx \quad \Rightarrow \quad \frac{1}{2}du = (e^x)dx$.
So we can then substitute:

$$\begin{aligned}\int \frac{e^x}{2+2e^x} dx &= \int \frac{1}{u}(\frac{1}{2}du) \\ &= \frac{1}{2}\int \frac{1}{u} du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|2+2e^x| + C\end{aligned}$$

Definite Integration and the Fundamental Theorem of Calculus

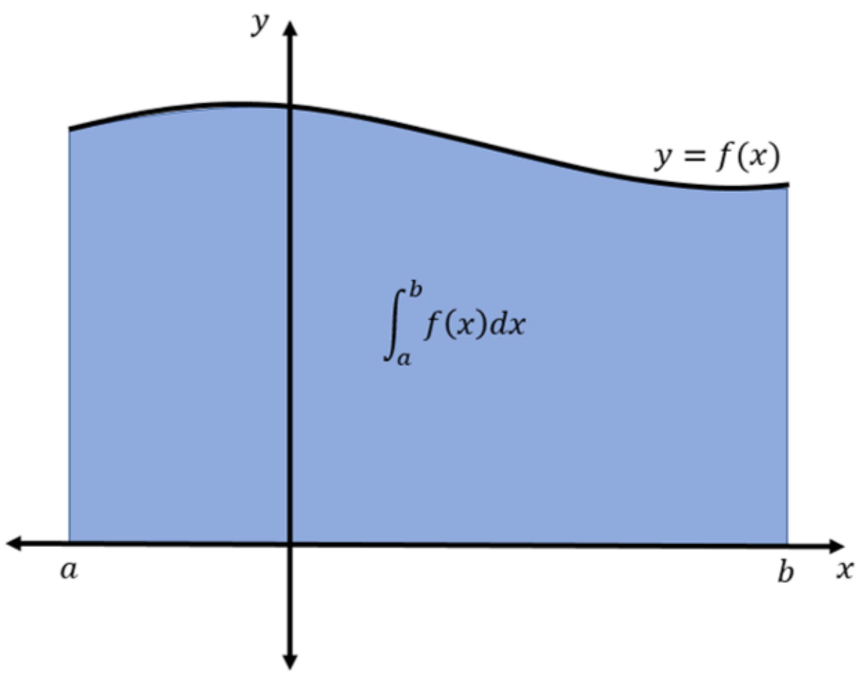
Indefinite integration gave you a *function* (e.g., $\int 2xdx = x^2 + C$). We now introduce the definite integral.

Fundamental Theorem of Calculus. Let $f(x)$ be a continuous function on an interval $[a, b]$ with an antiderivative $F(x)$ (i.e., $F'(x) = f(x)$). The **definite integral** of $f(x)$ from a to b is defined as

$$\int_a^b f(x)dx = F(b) - F(a).$$

a is called the **lower limit** and b is called the **upper limit**. The fundamental theorem of calculus simply says "the integral of $f(x)$ from a to b is $F(b) - F(a)$".
**note: the fundamental theorem of calculus is more formally stated in the Riemann sums section.*

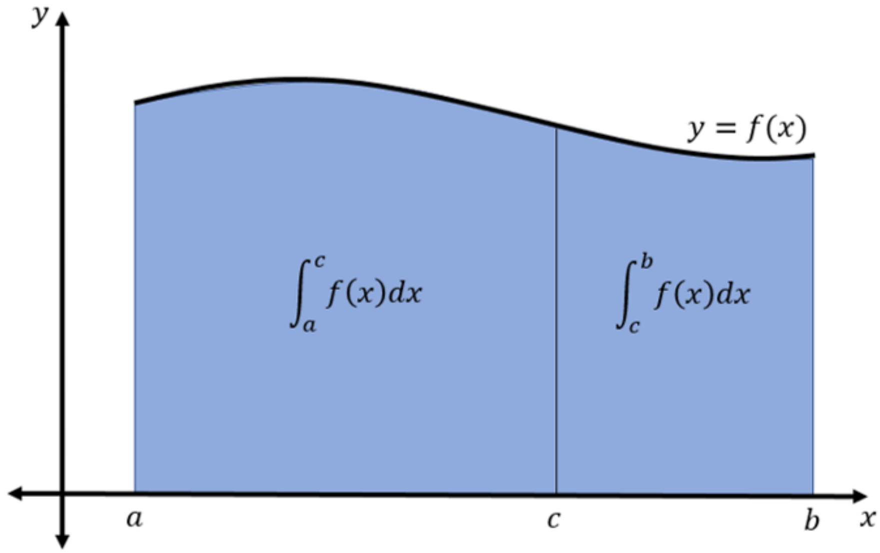
The definite integral of $f(x)$ from $x = a$ to $x = b$ can visually interpreted as "the region between the $x-$axis and the curve $y = f(x)$, as shown in the figure below.



There are two important theorems:

Theorem 1.
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad \text{where } a \le c \le b$$

In other words, the integral from a to b is equal to the integral from a to c, plus the integral from c to b, as shown in the figure below.



Theorem 2.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

In other words, if we "flip" the limits of integration, then we must change the sign of the definite integral (i.e., multiply by -1).

We now try to present at least one example for each type of problem you might expect to encounter in this course. However, surely the problems you might be asked to solve are not limited to this short list of examples.

Example 24.1. Find the area under the curve $f(x) = \begin{cases} x+1, & x \leq 1 \\ 3-x, & x \geq 1 \end{cases}$, from $x = 0$ to $x = 2$.

**note: piecewise functions can be found in Appendix I.*

$$\begin{aligned}
\int_0^2 f(x)dx &= \int_0^1 f(x)dx + \int_1^2 f(x)dx \\
&= \int_0^1 (x+1)dx + \int_1^2 (3-x)dx \\
&= \left[\frac{x^2}{2} + x\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 \\
&= \left(\frac{1}{2} + 1\right) + \left(6 - 2\right) - \left(3 - \frac{1}{2}\right) \\
&= 3
\end{aligned}$$

Example 24.2. Find the area under the curve $y = (x-3)^4$ from $x = 1$ to $x = 4$.

$$\int_1^4 (x-3)^4 dx = ?$$

We require $u-$substitution. Let $u = x - 3$ and so $du = dx$. Moreover, $x = 1 \to u = -2$ and $x = 4 \to u = 1$. Substituting in,

$$\int_{x=1}^{x=4} (x-3)^4 dx = \int_{u=-2}^{u=1} u^4 du = \frac{u^5}{5}\Big|_{u=-2}^{u=1} = \frac{1^5}{5} - \frac{(-2)^5}{5} = \frac{1}{5} - \frac{-32}{5} = \frac{33}{5}$$

**note: Alternatively, if you want to plug in the (original) limits of integration $x = 1$ and $x = 4$, then you must plug u in terms of x back into the answer.*

$$\int_{x=1}^{x=4} (x-3)^4 dx = \frac{(x-3)^5}{5}\Big|_{x=1}^{x=4} = \frac{1}{5} - \frac{(-2)^5}{5} = \frac{33}{5}$$

Example 24.3. Given $\int_3^1 f(x)dx = -3$ and $\int_1^3 g(x)dx = -1$, find $\int_1^3 (2f(x) - g(x))dx$.

$$\begin{aligned}
&= \int_1^3 (2f(x) - g(x))dx \\
&= \int_1^3 (2f(x))dx - \int_1^3 g(x)dx \\
&= 2(+3) - (-1) = 7
\end{aligned}$$

Example 24.4. A company's marginal cost function is $0.1x^2 - x + 12$ (in thousands of dollars per unit), where x denotes the number of units produced in 1 day. Determine the increase in cost if the production level is raised from $x = 1$ to $x = 3$ units.

We know that $C'(x) = 0.1x^2 - x + 12$, and we want to know $C(3) - C(1)$ (i.e., the increase in cost from $x = 1$ to $x = 3$. By the fundamental theorem of calculus,

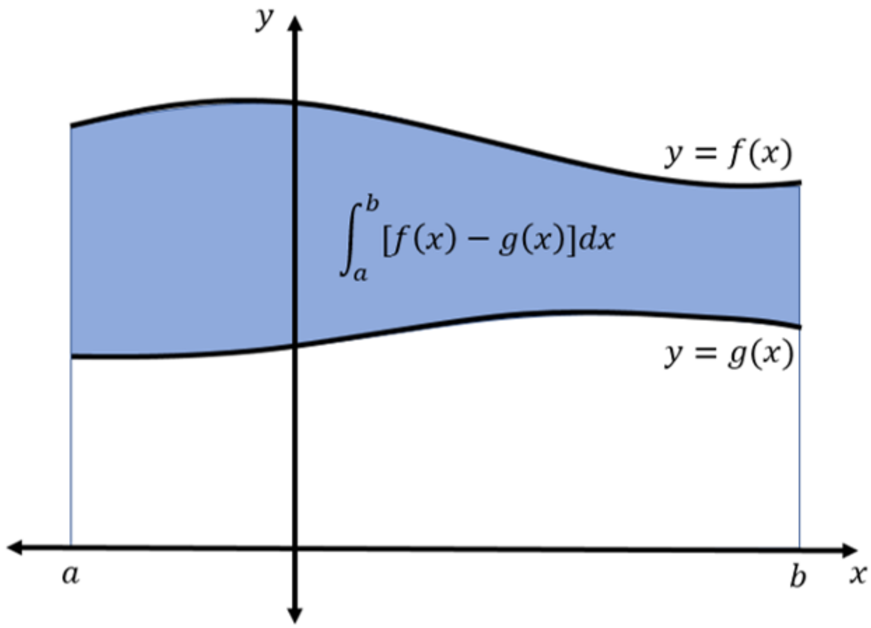
$$\begin{aligned}
C(3) - C(1) &= \int_1^3 C'(x)dx \\
&= \int_1^3 (0.1x^2 - x + 12)dx \\
&= \frac{1}{30}x^3 - \frac{x^2}{2} + 12x\Big|_1^3 \\
&= \left(\frac{3^3}{30} - \frac{3^2}{2} + 12 \cdot 3\right) - \left(\frac{1^3}{30} - \frac{1^2}{2} + 12 \cdot 1\right) \\
&= \left(\frac{27}{30} - \frac{9}{2} + 36\right) - \left(\frac{1}{30} - \frac{1}{2} + 12\right) \\
&= \frac{26}{30} - \frac{8}{2} + 24 \\
&= \frac{13}{15} + 20 \\
&= \frac{13}{15} + \frac{300}{15} \\
&= \frac{313}{15} \text{ thousands of dollars}
\end{aligned}$$

Definite Integration: Regions Between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over the interval $[a, b]$ with $f(x) \geq g(x)$ for all $a \leq x \leq b$ (i.e., $f(x)$ is always *above* $g(x)$). Then, the **region between the curves** $y = f(x)$ **and** $y = g(x)$ **from** a **to** b is defined as

$$\int_a^b [f(x) - g(x)]dx,$$

which can be visualized as in the figure below.



Example 25.1. Find the area bounded by $f(x) = -x^2 + 6x - 5$ and $g(x) = 2x - 5$.

First, set them equal and solve for the points where the two curves meet:
**we must do this step between one curve might not always be "above" the other.*

$$-x^2 + 6x - 5 = 2x - 5$$
$$-x^2 + 4x = 0$$
$$x(-x + 4) = 0$$
$$x = 0 \text{ and } x = 4$$

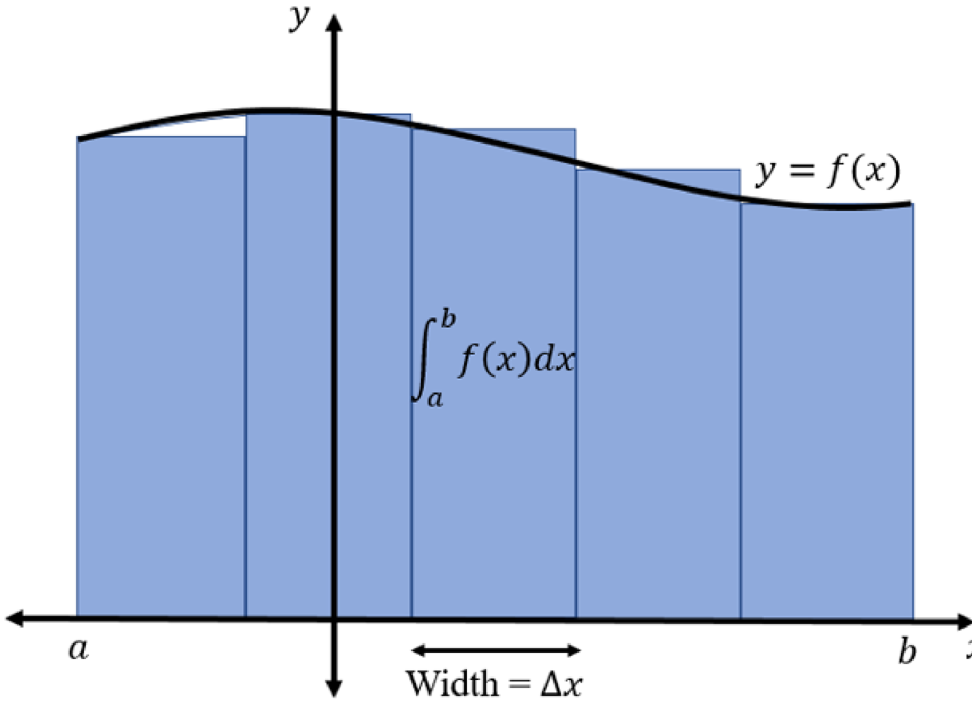
Now test to see which function is the one on top in the interval $[0, 4]$.

$$@x = 2, f(2) = -(2)^2 + 6(2) - 5 = +3$$
$$@x = 2, g(2) = 2(2) - 5 = -1$$

$$
\begin{aligned}
A &= \int_0^4 ((-x^2+6x-5)-(2x-5))dx \\
&= \int_0^4 (-x^2+4x)dx \\
&= -\frac{1}{3}x^3 + 2x^2|_0^4 \\
&= -\frac{1}{3}(4)^3 + 2(4)^2 - (-\frac{1}{3}(0)^3 + 2(0)^2) \\
&= -\frac{64}{3} + 32 - (0+0) = \frac{32}{3} \approx 10.667
\end{aligned}
$$

Riemann Sums

As the figure below shows, we can try approximating the definite integral of $f(x)$ from a to b using rectangles. The more rectangles we use, the better the approximation should get. This is what we refer to as **Riemann sums**.



Notice that in the figure above, the width of each rectangle is Δx and the height of each rectangle is $f(x)$ evaluated at the *left boundary* of each interval. We call this the Riemann sum with the **Left Endpoint Rule**. We could have the height be $f(x)$ evaluated at the *right boundary* of each interval using the **Right Endpoint Rule**. Or, we could use the midpoint of each interval using the **Midpoint Rule**. In fact, we can use *any* value in each interval.

Let $f(x)$ be a continuous function over the interval $[x_0 = a, b = x_n]$; n be the number of intervals; Δx be the width of *each* interval; $[x_{i-1}, x_i]$ be the n-intervals.

(General) Riemann Sum	$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$
Left Endpoint Rule	$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$
Right Endpoint Rule	$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)\Delta x$
Midpoint Rule	$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\Big(\frac{1}{2}(x_{i-1} + x_i)\Big)\Delta x$

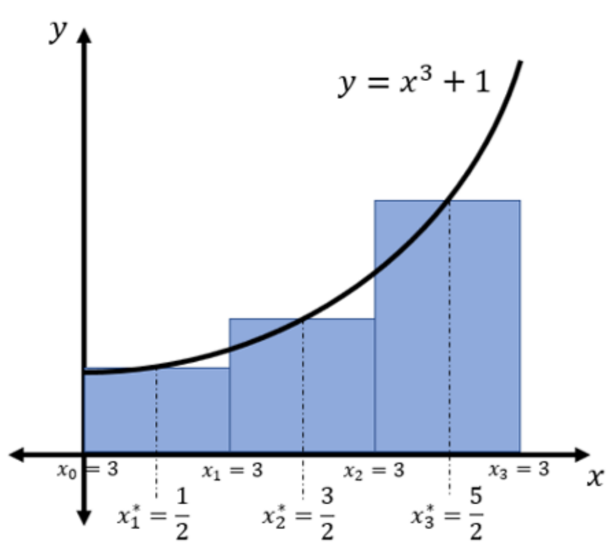
**note: the ONLY difference between all of these Riemann sums is the "test" points, x_i^*.*

Fundamental Theorem of Calculus (formally). Let $f(x)$ be a continuous function on an interval $[a, b]$ with an antiderivative $F(x)$ (i.e., $F'(x) = f(x)$). If $\Delta x = \dfrac{b-a}{n}$, where n is the number of sub-intervals, then

$$F(b) - F(a) = \int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Example 26.1. Using the midpoint rule with the given value of $n = 3$ (i.e., three sub-intervals/"bins"), approximate the integral

$$\int_0^3 (x^3 + 1)dx$$



As seen in the figure above, the “test points” are $x_1^* = \frac{1}{2}$, $x_2^* = \frac{3}{2}$, and $x_3^* = \frac{5}{2}$.

$f(x_1^*) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 1 = \frac{9}{8}$, $\quad f(x_2^*) = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 + 1 = \frac{35}{8}$, $\quad f(x_3^*) = f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^3 + 1 = \frac{133}{8}$

Since $\Delta x = 1$ for each interval, using the midpoint rule

$$\int_0^3 (x^3 + 1)dx \approx \frac{9}{8} \cdot 1 + \frac{35}{8} \cdot 1 + \frac{133}{8} \cdot 1 = \frac{177}{8}$$

Appendix A: Domain and Range

Domain: The values a function evaluates (e.g., the *independent variable*).
Range: All possible values of a function (e.g., the *dependent variable*).

Here are a few important functions.

(a) $f(x) = e^x$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$

(b) $f(x) = \ln x$
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

(c) $f(x) = \dfrac{1}{x}$
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

(d) $f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Appendix B: Synthetic Division

We use synthetic division to factor and find the roots of polynomials with the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We could technically use long-division (and you certainly can if you prefer), but as the name implies, it is *long*, tedious, and we'd rather not do it.

The steps for synthetic division are best explained using an example.

Example 28.1. Given the polynomial $y = 12x^3 + 16x^2 + 3x - 1$, put the polynomial in factored form and find the corresponding roots.

Step 1.
Find all possible factors of $a_n a_0$ (i.e. the first and last coefficients multiplied together). The roots of the polynomial are going to be among those factors.

In our case, $a_n a_0 = (12)(-1) = -12$. So the set of all possible factors of -12 is $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. All we need to keep going is ONE factor. To do this, simply plug possible roots from our set into our polynomial until you get a root. Let's start by plugging in $x = +1 \Rightarrow 12(1)^3 + 16(1)^2 + 3(1) - 1 = 30 \neq 0 \Rightarrow x = +1$ is not a root. Okay, let's try $x = -1 \Rightarrow 12(-1)^3 + 16(-1)^2 + 3(-1) - 1 = 0 \Rightarrow x = -1$ is a root! Fantastic! Now, we can use $x = -1$ and proceed.

Step 2.
We use the table notation seen below. Place our *known root* x_0 in the first row, on the far left. Fill in the rest of the first row with the coefficients of the polynomial (BE CAREFUL not to forget zeros if applicable... e.g. $2x^2 + 3 = 2x^2 + 0x + 3$). Then, bring down the first coefficient.

$$\begin{array}{r|rccc} x_0 & a_n & \cdots & a_1 & a_0 \\ \hline & & & & \\ \hline & a_n & & & \end{array}$$

For our example, the synthetic division table is given by

$$\begin{array}{r|rrrr} -1 & 12 & 16 & 3 & -1 \\ \hline & & & & \\ \hline & 12 & & & \end{array}$$

Step 3.
Multiple the first coefficient that you brought down (a_n) with the root (x_0), and write it directly below the second coefficient. Then, add the elements in the third column and write the sum directly below in the third row.

$$\begin{array}{r|rrrr} -1 & 12 & 16 & 3 & -1 \\ \hline & & -12 & & \\ \hline & 12 & 4 & & \end{array}$$

Step 4.
Continue this pattern of multiplying the next number in the third row with the root...

$$\begin{array}{r|rrrr} -1 & 12 & 16 & 3 & -1 \\ \hline & & -12 & -4 & 1 \\ \hline & 12 & 4 & -1 & 0 \end{array}$$

If you did everything *correctly*, then you should get a zero for the last element in the third row. If you did not get a zero, then either your algebra is incorrect, or you did not choose an actual root of the polynomial.

The numbers in the third row correspond to the coefficients of the next polynomial in the factored form. For instance, we can write the original polynomial as $y = (x - (-1))(12x^2 + 4x - 1)$. Observe that the coefficients of the polynomial $12x^2 + 4x - 1$ are the same as the third row!

Step 5.
Repeat the synthetic division process for the new polynomial, which for us is $12x^2 + 4x - 1$. **OR** if it is a quadratic polynomial, factor with the quadratic formula or whichever method you prefer. The end result for this example ends up being

$$y = (x + 1)(6x - 1)(2x + 1).$$

Appendix C: Absolute Values

Absolute values force the range to be nonnegative. We first find all the regions where a function $f(x)$ is negative, and make the range positive. For instance,

$$|x| = \begin{cases} -x, & x \leq 0 \\ x, & x \geq 0. \end{cases}$$

$$|x^2 - 4| = \begin{cases} x^2 - 4, & (-\infty, -2] \cup [2, \infty) \\ -(x^2 - 4), & [-2, 2] \end{cases}$$

**note: for piecewise function, see Appendix I.*

Appendix D: Laws of Logarithms

First, a few important values: $\ln(1) = 0$, $\ln(e) = 1$, and $\ln(0) =$undefined. **recall the graph of* $\ln x$.

Here are the laws of logarithms:

$$\log(ab) = \log(a) + \log(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$a\log(b) = \log(b^a)$$

$$\log_a(b) = x \Leftrightarrow a^x = b$$

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

Appendix E: Laws of Exponents

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$(ab)^x = a^x b^x$$

$$x^{-a} = \frac{1}{x^a}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$\sqrt[a]{x^b} = x^{\frac{b}{a}}$$

Appendix F: Leibniz Notation

Leibniz notation for the derivative of a function $f(x)$ is

$$\frac{df}{dx} \quad \text{or rather,} \quad \frac{d}{dx}(f).$$

We note that $\frac{d}{dx}$ means "take a derivative with respect to x." Similarly, $\frac{d}{dt}$ means "take a derivative with respect to t." All other variables that we are not taking a derivative with respect to, are treated as constants.

Example 32.1. Given $f = t^3 + 4xt + e^s$, evaluate the following:

$$\text{(a) } \frac{df}{dx}, \quad \text{(b) } \frac{df}{dt}, \quad \text{(c) } \frac{df}{ds}.$$

(a) Take a derivative with respect to x and treat t, s as constants.

$$\frac{df}{dx} = 0 + 4t + 0 = 4t.$$

(b) Take a derivative with respect to t and treat x, s as constants.

$$\frac{df}{dt} = 3t^2 + 4x + 0 = 3t^2 + 4x.$$

(c) Take a derivative with respect to s and treat x, t as constants.

$$\frac{df}{ds} = 0 + 0 + e^s = e^s.$$

Furthermore, as per notation for higher order derivatives, we write

Second derivative: $\left(\frac{d}{dx}\right)^2(f) = \frac{d^2}{dx^2}(f) = \frac{d^2 f}{dx^2}$

Third derivative: $\left(\frac{d}{dx}\right)^3(f) = \frac{d^3}{dx^3}(f) = \frac{d^3 f}{dx^3}$

Fourth derivative: $\left(\frac{d}{dx}\right)^4(f) = \frac{d^4}{dx^4}(f) = \frac{d^4 f}{dx^4}$

Appendix G: Finding Units

The reason we like Leibniz notation is because it makes determining units much easier. For instance, the units of the first derivative $f'(x)$ are "units of f"/"units of x," which is more clearly seen from Leibniz notation

$$\frac{df}{dx} \leftrightarrow \frac{[\text{units of } f]}{[\text{units of } x]}$$

Example 33.1. Consider a bank account whose money (in dollars) is increasing with respect to time (in years). This can be modeled with the equation $f(t)$.

Derivative	Units
$\frac{df}{dt}$	$\frac{dollars}{year}$
$\frac{d^2f}{dt^2}$	$\frac{dollars}{(year)^2}$
$\frac{d^3f}{dt^3}$	$\frac{dollars}{(year)^3}$

Example 33.2. Consider a car whose distance (in meters) is changing with respect to time (in seconds). This can be modeled with the function $f(t)$.

Derivative	Units
$\frac{df}{dt}$	$\frac{m}{s}$
$\frac{d^2f}{dt^2}$	$\frac{m}{s^2}$
$\frac{d^3f}{dt^3}$	$\frac{m}{s^3}$

Appendix H: Area, Volume and Perimeter Equations

Rectangle

$$A = lw \text{ (area)}, \qquad P = 2l + 2w \text{ (perimeter)}$$

Box

$$V = lwh \text{ (volume)}, \qquad SA = 2lw + 2lh + 2wh \text{ (surface area)}$$

Circle

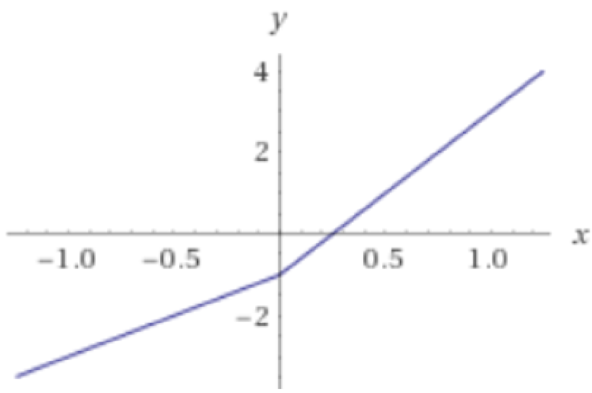
$$A = \pi r^2 \text{ (area)}, \qquad C = 2\pi r \text{ (circumference)}$$

Cylinder

$$V = h\pi r^2 \text{ (volume)}, \qquad SA = 2\pi r^2 + 2\pi rh \text{ (surface area)}$$

Appendix I: Piecewise Functions

If a curve is composed of several different functions (e.g., it takes on one function for half the domain and another function for the other half), then we use piecewise functions. For example,



$$f(x) = \begin{cases} 2x - 1, & -\infty < x \leq 0 \\ 4x - 1, & 0 \leq x < \infty \end{cases}$$

Example 35.1. Consider the function provided above. It is says that for $-\infty < x \leq 0$, the $y-$values are $2x - 1$; and for $0 \leq x < \infty$, the $y-$values are $4x - 1$.

Another way to think of this is that $y = 2x - 1$ is the graph to the *left* of $x = 0$; and $y = 4x - 1$ is the graph to the *right* of $x = 0$.

Appendix J: Equations in Economics

The two majors equations you need to know are

$$\boxed{P(x) = R(x) - C(x)}$$

and

$$\boxed{R(x) = x \cdot p(x)}$$

where $P(x)$ is the *profit*; $R(x)$ is the *revenue*; $C(x)$ is the *cost*; $p(x)$ is the demand equation for the *price*; and x is the *amount of a certain commodity.*

Marginal Profit, Revenue and Cost:

$$\boxed{\text{Marginal Profit} = P'(x), \quad \text{Marginal Revenue} = R'(x), \quad \text{Marginal Cost} = C'(x)}$$

**for simplicity, think "derivative" when you see the word "marginal."*

Average Profit, Revenue and Cost:

$$\boxed{\text{Average Profit} = \frac{P(x)}{x}, \quad \text{Average Revenue} = \frac{R(x)}{x}, \quad \text{Average Cost} = \frac{C(x)}{x}}$$

**these equations make sense...if you want the average of something, you divide by however many "things" you possess.*

Appendix K: Other Important Equations

Quadratic Formula for $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Pythagorean Theorem for Right Triangles

$$a^2 + b^2 = c^2, \quad \text{where } c \text{ is the hypotenuse}$$

Relation Between Distance, Velocity, and Acceleration

$$s(t) = \text{distance}$$

$$v(t) = \frac{ds}{dt} = \text{velocity}$$

**velocity is the derivative of distance.*

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \text{acceleration}$$

**acceleration is the derivative of velocity (i.e., second derivative of distance).*

Equation for a Circle

$$x^2 + y^2 = r^2, \quad \text{where } r \text{ is the radius}$$

Equation for an Ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \quad \text{where } a,\ b \text{ are the radii in } x,\ y \text{ directions, respectively.}$$