APPLIED MATHEMATICS PRELIM PREP 2021
WEEK 0 - REVIEW WORKSHEET

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The purpose of this review worksheet is to make sure your *fundamentals* are in place. You will not actually solve many differential equations in this worksheet; that will be the purpose of the actual prelim prep worksheets. That being said, this worksheet is not completely reminiscent of what prelim problems will look like (although I wouldn't be shocked if one or two of these problems popped up on a prelim). Please complete these problems before we start our first official prelim prep session. Most questions that you might have can be answered by "Applied Partial Differential Equations with Fourier Series and Boundary Value Problems" by Professor Richard Haberman. You should buy this book or find an electronic copy; the fourth edition –although an older edition– is practically the same as the current edition, but costs much less. I am also available via text/Zoom to answer questions.

Method of Undetermined Coefficients; Variation of Parameters

Several problems on the prelim will require you to solve nonhomogeneous second order ordinary differential equations. It is imperative that you know these two techniques like the back of your hand. Otherwise, you will not get very far with several prelim problems. As my first PDEs professor once said,

"How can I expect you to solve a basic PDE when you can't even solve a basic ODE?"

Exercise 1. Find the general solutions of the following ODEs using the method of undetermined coefficients.

(a) $y'' - 2y' - 3y = 3e^{2t}$
(b) $y'' - y' - 2y = -2t + 4t^2$
(c) $y'' - 2y' - 3y = -3te^{-t}$
(d) $y'' + y = 3\sin(2t) + t\cos(2t)$
(e) $y'' + y = \cosh(x)$

Exercise 2. Find the general solutions of the following ODEs using variation of parameters.

(a) $y'' - 5y' + 6y = t^9$
(b) $y'' + y = \tan(x)$, $0 < t < \pi/2$
(c) $4y'' + y = 2\sec(t/2)$, $0 < t < \pi$
(d) $y'' + 4y = 3\csc(x)$
(e) $y'' - 2y' + y = e^t/(1 + t^2)$

Fourier Series

At the heart of several problems that we solve on the applied math prelim lies Fourier series. It's important to know how to find the Fourier series of a given signal/function.

The traditional Fourier series of a function $f(x)$ that is periodic on the interval $[-L, L]$ can be expressed by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\},$$

where for all $n \in \mathbb{N}$ (including $n = 0$)

$$a_n = \frac{1}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad b_n = \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Exercise 3. Compute the Fourier series of the saw-tooth function over a single period $[-\pi, \pi]$,

$$f(x) = x, \qquad -\pi < x < \pi.$$

Exercise 4. Compute the Fourier series of the following function over a single period $[-\pi, \pi]$,

$$f(x) = e^{-x}, \qquad -\pi < x < \pi.$$

The traditional Fourier series (over $[-\pi, \pi]$) can also be expressed by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\} = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Exercise 5. Find the relationship between a_n, b_n, and c_n (all three). Further find the relationship between c_n and c_{-n}. Then, show that the coefficients c_n are defined by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

**note – the series involving coefficients c_n is the complex Fourier series.*

When a function $f(x)$ is only defined on $[0, L]$, it is common to extend this to an even or odd function $F(x)$ defined on $[-L, L]$. Thus, the Fourier series of $F(x)$ on the interval $[0, L]$ will give us the desired Fourier series of $f(x)$. The even extension $F_e(x)$ gives us the **Fourier cosine series**; and the odd extension $F_o(x)$ gives us the **Fourier sine series**.

Exercise 6. Given a function $f(x)$ defined only on $[0, L]$, perform an even extension $F_e(x)$ so that we are defined on $[-L, L]$. Find the Fourier series of $F_e(x)$ in order to find the Fourier cosine series of $f(x)$.

**note – By construction, $F_e(x)$ is an even function. Thus, we do not expect any sine terms in our series.*

Exercise 7. Given a function $f(x)$ defined only on $[0, L]$, perform an odd extension $F_o(x)$ so that we are defined on $[-L, L]$. Find the Fourier series of $F_o(x)$ in order to find the Fourier sine series of $f(x)$.

**note – By construction, $F_o(x)$ is an odd function. Thus, we do not expect any cosine terms in our series.*

Question to ponder. If a function defined on $[-L, L]$ is already an even function, then what can you say about the coefficients b_n? Similarly, if the function is already an odd function, then what can you say about the coefficients a_n?

Change of Coordinates/Variables

The purpose of this section is to sharpen your algebra mechanics, particularly when changing variables. We will solve PDEs in these coordinate systems later on.

Exercise 8. The Laplacian of a function u in Cartesian coordinates is

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Write Δu in cylindrical coordinates (i.e., polar coordinates in the xy-plane).

Exercise 9. Consider the ordinary differential equation (with $n > 0$)

$$(r^2 R')' + (k^2 r^2 - n(n+1))R = 0.$$

Use the change of variables $R(r) = Z(r)/\sqrt{r}$ to get Bessel's differential equation of order $n + 1/2$,

$$Z'' + \frac{1}{r}Z' + \left(k^2 - \frac{1}{r^2}\left(n + \frac{1}{2}\right)^2\right)Z,$$

for which the solution is $Z(r) = J_{n+1/2}(kr)$.

Integral Transforms

For the sake of this review sheet (and my prelim prep sessions), the Fourier transform pair will be defined as

$$\hat{f}(k) = \mathcal{F}\{f(x)\}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\}(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

Other definitions of the (inverse) Fourier transform are OKAY, but note that several theorems (e.g., shift theorem, convolution theorem) will be slightly different. Just note that this worksheet and those thereafter shall use the definitions above.

Exercise 10. Prove the following properties of the Fourier transform.

(a) $\mathcal{F}\{e^{-a|x|}\} = \dfrac{a}{\pi(a^2+k^2)} \qquad (a > 0)$

For the following problems, assume $\mathcal{F}\{f(x)\} = g(k)$.

(b) $\mathcal{F}\{g(-x)\} = \dfrac{f(k)}{2\pi}$

(c) $\mathcal{F}\{f'(x)\} = +ikg(k)$

(d) $\mathcal{F}\{-ixf(x)\} = g'(k)$

(e) $\mathcal{F}\{f(x-\alpha)\} = e^{-ik\alpha}g(k)$

Exercise 11. Given the definition of the Fourier transform as stated before, assume

$$f(x) \leftrightarrow \hat{f}(k) \qquad \text{and} \qquad g(x) \leftrightarrow \hat{g}(k).$$

Then, the **convolution theorem** for the Fourier transform states

$$\frac{1}{2\pi}(f * g)(x) \leftrightarrow (\hat{f}\hat{g})(k),$$

where the convolution of $f(x)$ and $g(x)$ is defined as $(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi$.

Furthermore, the **Fourier transform of a Gaussian function** is

$$e^{-ax^2} \leftrightarrow \frac{1}{\sqrt{4\pi a}}e^{-k^2/(4a)}, \qquad a > 0.$$

Using the properties of Fourier transforms given (including exercise 8), find the following.

(a) Find the inverse Fourier transform of $\hat{f}(k)e^{-a|k|}$, where $a > 0$.

(b) Show that $\mathcal{F}\{f(\alpha x - \beta)\} = \alpha e^{-ik\beta/\alpha}\hat{f}(k/\alpha)$.

(c) Find the inverse Fourier transform of $\dfrac{1}{2\pi}e^{-Dk^2t}$, where constants $D, t > 0$.

(d) Find the Fourier transform of $xe^{-|x|}$. Do not do this directly; use the properties given.

Exercise 12. For this problem, use the properties already stated (i.e., do not do this problem directly). Let $u(x) = e^{-|x|}$. Find the function $f(x)$ (in terms of $u(x)$) such that $\hat{f}(k) = \dfrac{2\pi}{(1+k^2)^2}$.

Similar to the Fourier sine and cosine series, there are Fourier sine and cosine transforms (that are used for signals/functions defined on the half-line). The Fourier sine and cosine transform pairs are respectively defined as

$$\hat{f}_s(k) = \frac{1}{\pi}\int_0^\infty f(x)\sin(kx)dx, \qquad f(x) = 2\int_0^\infty \hat{f}_s(k)\sin(kx)dx$$

$$\hat{f}_c(k) = \frac{1}{\pi}\int_0^\infty f(x)\cos(kx)dx, \qquad f(x) = 2\int_0^\infty \hat{f}_c(k)\cos(kx)dx$$

Exercise 13. Let $f(x)$ be a function defined for $x > 0$. Perform an odd extension $F_o(x)$ and apply the Fourier transform to derive the Fourier sine transform of $f(x)$; note you'll have a factor of i floating around. Then, apply the inverse Fourier transform (note you'll have only integrated $0 < x < \infty$) to get the inverse Fourier sine transform.

Exercise 14. Let $f(x)$ be a function defined for $x > 0$. Perform an even extension $F_e(x)$ and apply the Fourier transform to derive the Fourier cosine transform of $f(x)$. Then, apply the inverse Fourier transform (note you'll have only integrated $0 < x < \infty$) to get the inverse Fourier cosine transform.

An important note. There are properties associated with the Fourier (co)sine transform similar to those for the Fourier transform. We are not going to deal with them in this worksheet. However, if given the properties you should be able to use them accordingly. In spirit, it is no different than dealing with the properties of the Fourier or Laplace transform.

Laplace transforms. It is safely assumed that you can take Laplace transforms. You'll be expected to use the Laplace transform to solve ordinary and partial differential equations on the prelim. Furthermore, you'll be expected to use properties of the Laplace transform similar to those of the Fourier transform. We will not focus on these things here. You practiced using integral transform properties with the Fourier transform. This studying should carry over when using the Laplace transform.

The 1D Wave Equation

The 1D wave equation over the real line $(-\infty, \infty)$ is

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x, t=0) = f(x), & x \in \mathbb{R}, \\ u_t(x, t=0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Exercise 15. Show that $u(x,t) = F(\xi) + G(\eta)$, where $\xi = x - ct$ and $\eta = x + ct$. In other words, the solution is a function of ξ plus a function of η.

Exercise 16. Derive D'Alembert's solution

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Exercise 17. Consider the 1D wave equation $u_{tt} - u_{xx} = 0$ (i.e., $c = 1$) defined only on the half-line $0 < x < \infty$. Draw the $xt-$plane; on this plane draw the line $\xi = x - t = 0$. Notice that there are two regions: (Region 1) $\xi(x,t) = x - t < 0$; and (Region 2) $\xi(x,t) = x - t > 0$.

(a) In which region can we directly use D'Alembert's solution? Justify using the domain of dependence (i.e., the characteristics $\xi(x,t)$ and $\eta(x,t)$).

(b) In the region where D'Almbert's solution cannot be directly applied, explain why not.

The Maximum/Minimum Principle for Heat Equation

The applied math prelim requires you to prove some very simple results. The worksheets to come will prepare you more in-depth. However, for the sake of review, you should be able to quickly prove the maximum/minimum principle for the 1D heat equation.

Exercise 18. Consider the 1D heat equation $u_t = u_{xx}$ in a region $(x,t) \in (0,1) \times (0,T]$. Let $\Gamma = (\{0,1\} \times [0,T]) \cup ([0,1] \times \{0\})$. Prove the maximum (minimum) principle for the 1D heat equation – if $u(x,t)$ satisfies the heat equation in $(0,1) \times (0,T]$, then the maximum (minimum) is attained on Γ.

Vector Calculus

The single-most important vector identity that you will use on the applied math prelim is

$$\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \nabla \cdot \mathbf{F},$$

where f is a "nice enough" scalar field and $\mathbf{F}$ is a "nice enough" vector field.

Exercise 19. We shall assume all fields are "nice enough" such that the following theorems can be applied. The divergence theorem over an open and bounded region $\Omega \subset \mathbb{R}^3$ states

$$\int_\Omega \nabla \cdot \mathbf{F} dV = \oint_{\partial\Omega} \mathbf{F} \cdot \hat{\mathbf{n}} dS.$$

Using the divergence theorem, derive Green's first identity,

$$\int_\Omega (\psi \nabla \cdot \mathbf{\Gamma} + \mathbf{\Gamma} \cdot \nabla \psi) dV = \oint_{\partial\Omega} \psi \mathbf{\Gamma} \cdot \hat{\mathbf{n}} dS.$$

Exercise 20. Let U be a bounded open subset of $\mathbb{R}^3$ with a "nice enough" boundary. If $u \in C_c^2(U)$ (i.e., twice continuously differentiable and of compact support on U), prove that

$$\int_U |\nabla u|^2 dV \leq \|u\|_{L^2(U)} \|\Delta u\|_{L^2(U)}.$$

**note – you'll need to use Hölder's inequality, but don't worry about needing to know Hölder's inequality on the applied math prelim. This purpose of this problem is just to give you practice using vector calculus.*